MST204.30

The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

Unit 30
Planetary orbits

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Prepared by the Course Team

The Open University

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Contents

Introduction

> I offer this work as the mathematical principles of philosophy, for the whole burden of philosophy seems to consist in this — from the phenomena of motions to investigate the forces of nature, and then from these forces to demonstrate the other phenomena; and to this end the general propositions in the first and second Books are directed. In the third Book I ... derive from celestial phenomena the forces of gravity with which bodies tend to the Sun and the several planets. Then from these forces, by other propositions which are also mathematical, I deduce the motions of the planets, the comets, the Moon, and the sea.
>
> Sir Isaac Newton, from the Preface to the First Edition of *Principia*.

The third book of Sir Isaac Newton's monumental work *Principia*, entitled *The System of the World*, was published in 1687, some twenty years after he had discovered many of its results. In it he establishes his universal law of gravitation and uses his three laws of motion to predict the character of the orbits of planets and their moons. Tradition has it that Halley, the astronomer, Hooke, the physicist, and Wren, the architect, had discussed at various times in 1684 the problem of determining the orbit of a planet around the Sun if the force between them varied as the inverse square of the distance. Unable to succeed with the problem, they took it to Newton, only to find that he had already solved it, together with other problems of planetary motion. Encouraged constantly by Halley, Newton finally published *Principia.* The theory which it expounded was extraordinarily successful. Only in the twentieth century, with the theories of relativity and quantum mechanics, has it become evident that there are natural processes not explained accurately by the Newtonian world-view.

The main aim of this unit is to apply Newtonian mechanics in order to derive Kepler's three laws of planetary orbits. Johann Kepler (1571–1630) arrived at these laws by trial and error, from careful analysis of the measurements of planetary orbits taken by himself and by Tycho Brahe (1546–1601), the Danish astronomer. The first two laws were published in 1609 and the third in 1619, so they precede Newton's mathematical derivations by a good lifetime.

In deriving the laws of planetary motion we shall use modern notation and methods, although it must be said that the calculus in its original form was the innovation of Newton and of his contemporary, Leibnitz. The modern use of geometric vectors was apparently an invention of J. Willard Gibbs (1839–1903), and is a major notational advance over the geometric methods which held sway in Newton's era.

In summary, then, there is little new mathematics introduced in this unit. Instead we utilize the methods of calculus and vector analysis developed in the course to describe one of the great scientific advances of the last three centuries.

Study guide

The material in Section 1 is largely revision, involving methods which will be of particular use later in the unit. A review of energy conservation for one-dimensional motion (from *Unit 7*) leads on to the corresponding topic for a particle moving in three dimensions. Then the planar motion of a particle is described in terms of polar coordinates and the associated unit vectors. This is a generalization of the treatment in *Unit 28*, where the particle motion was confined to a circle.

The time spent on Section 1 will depend upon your confidence with the previous mechanics in the course, but you should not spend more than 30% of your total study time here. The major part of your effort should be reserved for Sections 2 and 3.

Kepler's laws for planetary orbits are stated and explained in Section 2. Following this, Newton's universal law of gravitation is introduced. In Section 3 the equation of gravitational orbits is derived, and it is shown how Kepler's laws follow as a consequence of Newtonian mechanics.

The television programme in Section 4 covers a mathematical development similar to that in Section 3, but with greater reference to non-gravitational examples. More benefit will probably be obtained from the programme if you have managed to complete most of Section 3 first, but it should still be of help if viewed at an earlier stage.

1 Review of energy and planar motion

The ultimate goal of this unit is to model the motions of planets, satellites and comets. To facilitate this study it is useful to review some aspects of Newtonian mechanics, and that is the purpose of Section 1.

Subsection 1.1 will remind you about the material in *Unit 7* concerning particle motions in one dimension for which the total mechanical energy is constant. In Subsection 1.2 the corresponding motions in three dimensions are considered, and it is shown that the total mechanical energy is conserved provided that the force acting on the particle is *conservative* in the sense of *Unit 26*. This conclusion applies also in one or two dimensions.

Unit 7 Section 3

Unit 26 Subsection 4.3

Subsection 1.3 shows how motion in the plane may be described in terms of polar coordinates r, θ and the associated unit vectors $\mathbf{e}_r, \mathbf{e}_\theta$. You saw circular motion expressed in this way in *Unit 28*. Following the derivation of kinematical formulas, there are some applications of Newton's second law to find equations of motion in this coordinate system.

Unit 28 Subsection 4.2

1.1 Mechanical energy in one dimension

Consider a particle of mass m, moving in one dimension under the action of a force. We choose the x-axis to coincide with the line along which the particle moves. Then, as you know from *Unit 7*, Newton's second law is $m\ddot{x} = F$, where F is the x-component of the force. In general, the force may depend upon the location x of the particle, its velocity $\dot{x}$, the time t, or any combination of these variables. We consider here the case where F depends *only* upon the particle's position x. (A special case of this is when F is constant.) A position-dependent force in one dimension is called *conservative* because, as you may recall from *Unit 7* and as we shall shortly verify anew, the total mechanical energy is constant (conserved) for such a force.

You may see a particle described elsewhere as either a 'point mass' or a 'mass point'.

The description 'conservative' was not used when energy was introduced in *Unit 7* Section 3. You will see in the next subsection that energy-conserving forces are also conservative in the sense of *Unit 26*.

In terms of the force $F = F(x)$, the *potential energy* $U = U(x)$ of the particle is defined by the equation

$$F(x) = -\frac{dU}{dx}, \tag{1}$$

or equivalently by

$$U(x) = -\int_{x_0}^{x} F(s)\, ds,$$

where x_0 is any conveniently chosen point.

The point x_0 is the *datum* of the potential energy function U, for which $U(x_0) = 0$. This point (or equivalently, the constant of integration arising from the indefinite integral for U) is usually chosen so as to achieve the simplest possible form for $U(x)$.

Exercise 1

(i) Find the potential energy $U(x)$ corresponding to the force $F(x) = k$, where k is a constant.

(ii) Find the potential energy $U(x)$ corresponding to the force $F(x) = -kx$, where k is a constant.

(iii) Find the force $F(x)$ corresponding to the potential energy $U(x) = k/x$ ($x > 0$), where k is a constant.

[*Solution on page 47*]

When it is possible to define a potential energy function then the sum of the *kinetic energy* $\frac{1}{2}m\dot{x}^2$ and the potential energy $U(x)$ is constant throughout the motion of the particle. The sum

$$E = \tfrac{1}{2}m\dot{x}^2 + U(x) \tag{2}$$

is called the *total mechanical energy* of the particle, and its constancy is expressed by the equation $\dot{E} = 0$. To establish that E is constant, we differentiate Equation (2) with respect to time t. Using the chain rule on the right-hand side, this leads to

$$\dot{E} = m\ddot{x}\dot{x} + \frac{dU}{dx}\dot{x} = \dot{x}\left(m\ddot{x} + \frac{dU}{dx}\right).$$

But the term in brackets vanishes since, by Newton's second law and Equation (1),

$$m\ddot{x} = F(x) = -\frac{dU}{dx}.$$

Hence we have $\dot{E} = 0$. The fact that E is constant for a conservative force often provides useful information about the particle motion. For example, consider the harmonic oscillator (that is, a particle attached to one end of a perfect spring whose other end is fixed). Measuring x from the equilibrium position of the particle, the force F is given by

$$F(x) = -kx,$$

where k is the stiffness of the spring. The potential energy function is therefore

$$U(x) = \tfrac{1}{2}kx^2 + C,$$

where C is a constant. Any value of C may be chosen here since the value of $F(x) = -dU/dx$ is unaffected by the choice. One often takes $C = 0$. Figures 1(a) and 1(b) below show the potential energy $U(x)$ for a negative constant C and for the more conventional choice $C = 0$ respectively.

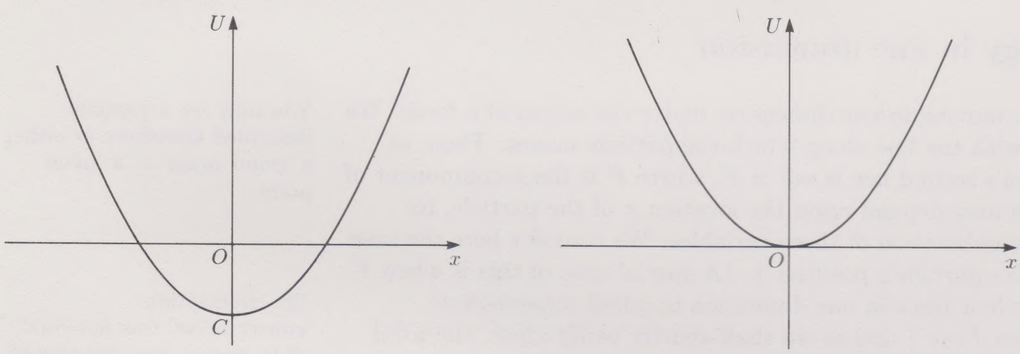


Figure 1

Either form for $U(x)$ gives the correct equation of motion. For any chosen value of C the total mechanical energy $E = \frac{1}{2}m\dot{x}^2 + U(x)$ is a constant, although its value depends on the choice of C.

To see how useful this knowledge is, suppose that initially, at $t = 0$, the particle has position x_0 and is moving with velocity v_0. This information establishes the total mechanical energy of the particle for all later times t, since $E(t) = E(0)$. Whatever value of C is chosen, we have

$$E = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}kx_0^2 + C = \tfrac{1}{2}m\dot{x}^2(t) + \tfrac{1}{2}kx^2(t) + C.$$

Solving for the speed squared gives

$$\dot{x}^2(t) = \frac{2}{m}\left(E - \tfrac{1}{2}kx^2(t) - C\right) = \frac{2}{m}\left(E - U(x)\right).$$

Since the left-hand side can never be negative, we know that the term in brackets on the right-hand side is also non-negative. Hence, for all times t we know that

$$U(x) \leq E.$$

This information is conveyed graphically in Figure 2, where we have chosen the value of C used in Figure 1(a). The constant value of E is represented by the horizontal dashed line. At each value of x the vertical distance between this horizontal line and the curve for $U(x)$ gives the kinetic energy $\frac{1}{2}m\dot{x}^2$ of the particle at that value of x. This difference, $E - U(x)$, does not depend on C. When the particle is located at $x = \alpha$ or at $x = \beta$, its kinetic energy is zero. The particle approaches these points, comes

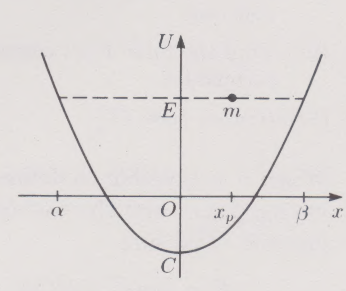


Figure 2

momentarily to a halt, and then reverses its direction of motion. For this reason, points like α and β are called *turning points*.

Note that the motion of the particle is periodic: at any point x_p between the turning points the speed is $\sqrt{2(E - U(x_p))/m}$, and this value is always reproduced when the particle returns to that point. The velocity at the point takes the values $\pm\sqrt{2(E - U(x_p))/m}$ alternately, depending on whether the particle is passing through x_p from left to right or from right to left. The particle passes through x_p twice in each complete cycle.

In fact, you know from the explicit solution for simple harmonic motion in *Unit 7* that in this case the particle oscillates sinusoidally between the turning points.

Unit 7 Subsection 2.3

Exercise 2

The position function for a particle of mass m, undergoing simple harmonic motion due to a perfect spring of stiffness k, can be written as

$$x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t,$$

where $\omega = \sqrt{k/m}$ is the angular frequency and x_0, v_0 are respectively the position and velocity of the particle at time $t = 0$. Verify explicitly that the total mechanical energy

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 + C$$

is constant, and find the turning points for this particular motion.

[*Solution on page 47*]

Explicit formula solutions to Newton's second law for the case of conservative forces cannot always be derived. If the potential energy function $U(x)$ is known, however, then the approach used prior to Exercise 2 may be applied. This gives qualitative information about the particle motion even when the position function can be established only by numerical computation. In all cases it is useful to know that E is constant. Consider, for example, the motion of a particle under the influence of the potential energy function shown in Figure 3.

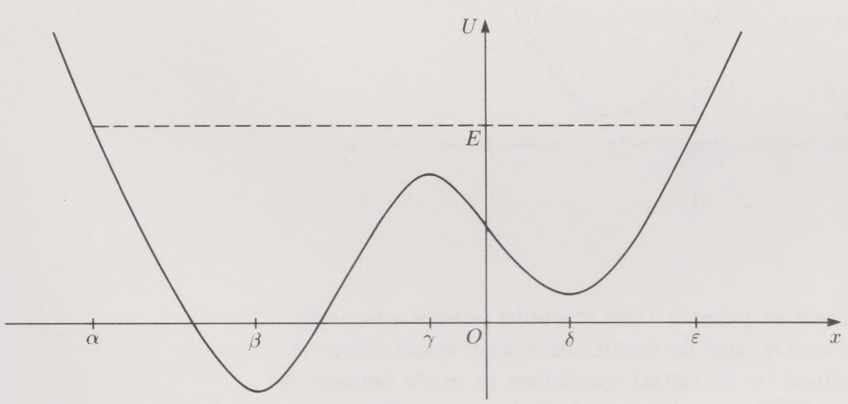


Figure 3

Given only the information provided by this graph, we can make the following statements about the particle's motion if it has the energy E indicated in the figure.

(i) The motion is periodic, and is confined between the turning points α and ε. At these points the particle slows to a stop and reverses its direction of motion.

(ii) At any point x for which $\alpha \le x \le \varepsilon$, the particle's velocity is $\pm\sqrt{2(E - U(x))/m}$.

(iii) The particle is moving fastest at $x = \beta$, since $E - U(x)$ has an overall maximum there.

(iv) As the particle passes through $x = \gamma$ in either direction, it slows down and then speeds up again. This follows from the fact that the speed $\sqrt{2(E - U(x))/m}$ has a local minimum there.

(v) As the particle passes through $x = \delta$ it experiences a local maximum of speed.

Example 1

What can be said about the motion of a particle under the influence of the potential energy function shown in Figure 4, if the energy E of the particle is as indicated on the figure? In particular, how many turning points are there at this energy level, what are these turning points and where does the particle move fastest?

The potential energy function of Figure 4 is the same as that in Figure 3, but the total mechanical energies of the particles are different.

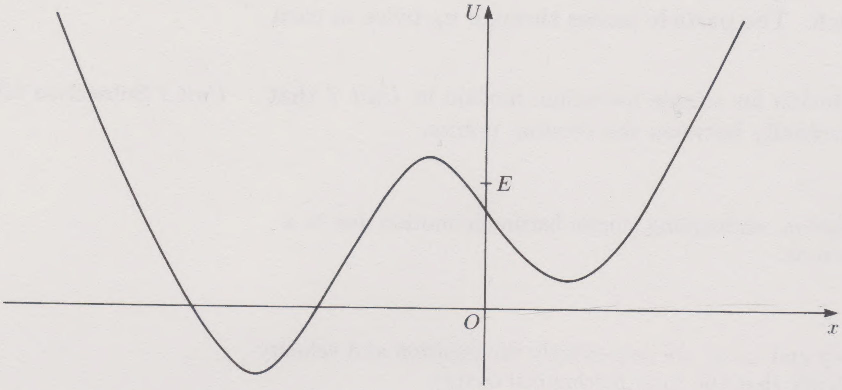


Figure 4

Solution

The particle is confined (see Figure 5) to periodic motion *either* between points α and γ *or* between points δ and ϕ, depending upon its initial position $x(0)$.

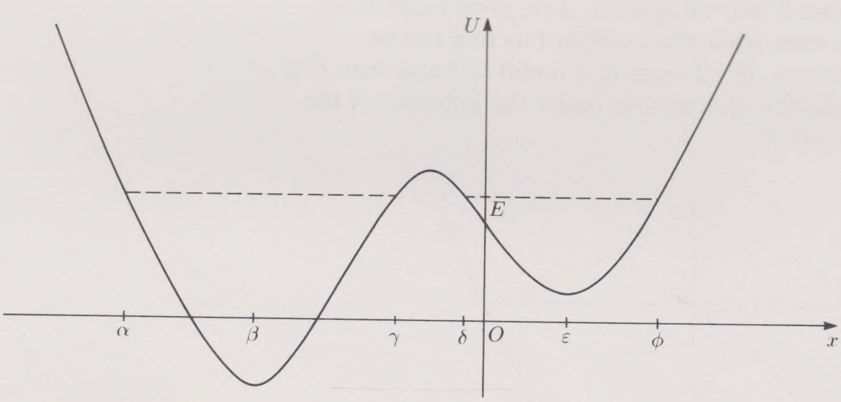


Figure 5

For instance, if it were released from rest at point α then it would pursue a periodic motion between the turning points α and γ, and its maximum speed would occur at $x = \beta$. Alternatively, it might be confined by its initial conditions to move between δ and ϕ, with maximum speed at $x = \varepsilon$. Motion at this energy E is not possible between points γ and δ, as the 'kinetic energy' $E - U(x)$ would be negative in this region. For the same reason, motion at the given energy is impossible for $x < \alpha$ or for $x > \phi$. □

It is also possible with such energy diagrams to consider the motion of a particle which is not bound to periodic motion. Such *unbound* motion occurs for the case depicted in Figure 6, where $U(x) = 0$ for $x \geq \delta$.

For example, suppose that the particle were released from rest at $x = \alpha$, which fixes its total mechanical energy for all subsequent times at $E = \frac{1}{2}m\dot{x}^2(0) + U(\alpha) = U(\alpha)$. The particle would then accelerate to the right until it picked up maximum speed at $x = \beta$. From $x = \beta$ to $x = \gamma$ it would decelerate to a local speed minimum at $x = \gamma$. It would then accelerate again to the right until it reached $x = \delta$, after which it would continue to the right with constant speed, in unbound motion. The speed is constant for $x > \delta$ because the potential energy in that region is constant (zero, in fact).

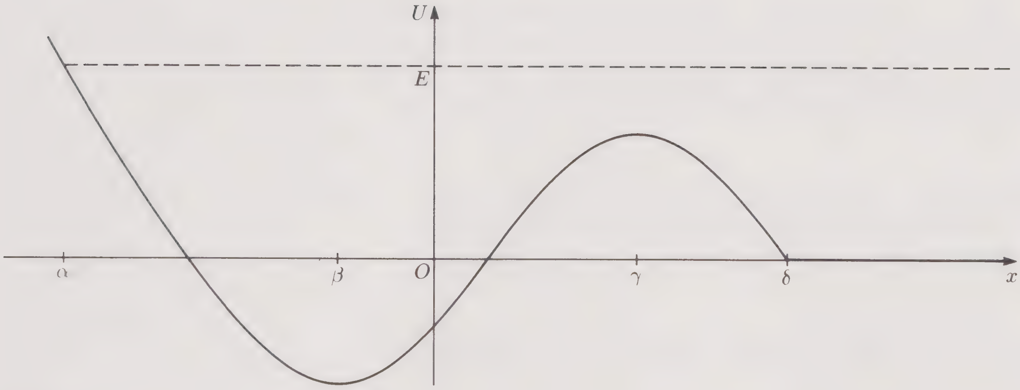


Figure 6

Exercise 3

For the particle of Figure 6, released from rest at $x = \alpha$, give in terms of m and U expressions for the particle's speed at $x = \beta$, $x = \gamma$ and for $x \geq \delta$. Arrange the speeds at the points β, γ and δ in order of magnitude.

Exercise 4

A particle is projected to the left in Figure 6 from some initial point x_0 to the right of δ, and with the total mechanical energy E shown in the figure. Give its initial velocity $\dot{x}(0)$ in terms of E, and describe its subsequent motion.

[*Solutions on page 47*]

1.2 Mechanical energy in three dimensions

Just as for one-dimensional motion, the force acting on a particle moving in two or three dimensions can depend upon position $\mathbf{r}$, velocity $\dot{\mathbf{r}}$ and time t, or on any combination of these variables. It often happens, however, that the force $\mathbf{F}$ depends only upon position, which may be denoted by writing $\mathbf{F} = \mathbf{F}(\mathbf{r})$ or $\mathbf{F} = \mathbf{F}(x, y, z)$. If *in addition* $\mathbf{F}$ can be expressed as the gradient of a scalar field, then it is called a **conservative force**. By convention the scalar field function here is written as $-U$, where $U = U(\mathbf{r}) = U(x, y, z)$ is called the **potential energy** of the particle. Thus in three (or fewer) dimensions, a conservative force $\mathbf{F}$ and potential energy U are connected by the equation

$$\mathbf{F} = -\operatorname{grad} U. \tag{3}$$

Note that in one dimension this reduces to

$$F\mathbf{i} = -\frac{dU}{dx}\mathbf{i} \quad \text{or} \quad F = -\frac{dU}{dx},$$

which is Equation (1) of the previous subsection.

The *gradient* of a scalar field ϕ, denoted by $\operatorname{grad}\phi$ or $\boldsymbol{\nabla}\phi$, was introduced in *Unit 26* Section 2. It is defined as the vector field

$$\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}.$$

Conservative vector fields were defined in terms of line integrals in *Unit 26* Section 3. In Subsection 4.3 of that unit it was shown that the definition is equivalent to the condition given here, involving the gradient of a scalar field.

Conservative forces were discussed briefly in *Unit 26* Subsection 4.3. It is enough for the purposes of this unit to recognize that where the force acting on the particle can be written as in Equation (3), the total mechanical energy E is constant, where

$$E = \tfrac{1}{2}m\dot{\mathbf{r}}^2 + U(x, y, z). \tag{4}$$

As pointed out in *Unit 26* Subsection 4.3, the *potential energy* U can be expressed as the line integral

$$-\int \mathbf{F} \cdot d\mathbf{r}.$$

For the one-dimensional case this reduces to

$$U = -\int F\, dx,$$

as in Subsection 1.1.

Here we have written $\dot{\mathbf{r}}^2$ as a shorthand for $\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$, which is also equal to $|\dot{\mathbf{r}}|^2$. In Equation (4) the kinetic energy $\frac{1}{2}m\dot{\mathbf{r}}^2$, the potential energy U and the total energy E are scalars. In order to show that E is constant, it suffices to establish that $\dot{E} = 0$. Differentiation of Equation (4) with respect to time t and use of the chain rule produces

$$\begin{aligned}
\dot{E} &= \frac{d}{dt}\left(\tfrac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + U(x, y, z)\right) \\
&= \tfrac{1}{2}m \times 2\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} \\
&= m\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \dot{\mathbf{r}} \cdot \operatorname{grad} U \\
&= \dot{\mathbf{r}} \cdot (m\ddot{\mathbf{r}} + \operatorname{grad} U).
\end{aligned}$$

The derivative of a scalar product was derived in *Unit 14* Subsection 4.3.

This form of the chain rule was introduced in *Unit 25* Subsection 2.4.

But by Equation (3), we have $\operatorname{grad} U = -\mathbf{F}$. Also $\mathbf{F} = m\ddot{\mathbf{r}}$ by Newton's second law, so the term in brackets vanishes and hence $\dot{E} = 0$. We have shown that, when the force has the form given by Equation (3), the total mechanical energy is constant. This outcome represents the natural extension to three dimensions of the results in Subsection 1.1.

Exercise 5

A particle of mass m moves under the influence of a uniform gravitational field, so that $\mathbf{F} = -mg\mathbf{j}$, where the unit vector $\mathbf{j}$ is directed vertically upwards. Show that this field is conservative by finding the corresponding potential energy function U.

Exercise 6

Given that $\mathbf{F}$ is a conservative force with potential energy $U = k/r$, where k is a constant and $r = (x^2 + y^2 + z^2)^{1/2}$, show that

$$\mathbf{F} = \frac{k}{r^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{k\mathbf{r}}{r^3}.$$

[*Solutions on page 47*]

There is a useful generalization to the result above concerning the constancy of E. Suppose that the particle is being acted upon by a conservative force, as defined by Equation (3), *plus* any force $\mathbf{G}$ which is always directed *perpendicular* to the particle's velocity. Then its equation of motion is

$$m\ddot{\mathbf{r}} = -\operatorname{grad} U + \mathbf{G},$$

where $\mathbf{G} \cdot \dot{\mathbf{r}} = 0$. In the exercise below you are asked to show that, under these more general circumstances, E is still constant.

Exercise 7

Suppose that a particle is acted upon by a net force $-\operatorname{grad} U + \mathbf{G}$, where $\mathbf{G} \cdot \dot{\mathbf{r}} = 0$. Show that if E is given by Equation (4) then $\dot{E} = 0$.

[*Solution on page 47*]

The result of Exercise 7 is often applied when modelling the constrained motion of objects. Consider, for instance, a particle of mass m sliding without friction down an inclined plane (see Figure 7). Since we are assuming that there is no friction present, the only forces acting are gravity and the reaction force $\mathbf{R}$. The latter is always perpendicular to the plane and hence also to the velocity. The gravitational force $-mg\mathbf{j}$ is conservative (as shown in Exercise 5) with potential energy function

$$U = mgy + C,$$

where C is any constant. If u is the speed then, from the result of Exercise 7, the total mechanical energy E is constant, where

$$E = \tfrac{1}{2}mu^2 + mgy + C.$$

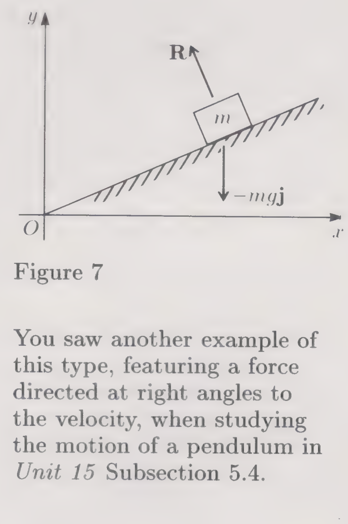


Figure 7

You saw another example of this type, featuring a force directed at right angles to the velocity, when studying the motion of a pendulum in *Unit 15* Subsection 5.4.

Exercise 8

In the situation just described, the particle is released from rest at a vertical height h above the x-axis in Figure 7. Find its speed u_0 when it reaches the origin.

[*Solution on page 48*]

1.3 Planar motion using polar coordinates

In this subsection we consider a means of describing the motion of a particle in a plane. For some applications it is convenient to describe the position vector $\mathbf{r}$ of the particle in terms of Cartesian coordinates (x, y) and Cartesian unit vectors $\mathbf{i}, \mathbf{j}$, that is,

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}. \tag{5}$$

Then the velocity and acceleration vectors are respectively

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} \tag{6}$$

and $$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t) = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j}. \tag{7}$$

The ease of derivation of Equations (6) and (7) from Equation (5) is due to the fact that $\mathbf{i}$ and $\mathbf{j}$ are constant vectors throughout the motion of the particle. In summary,

then, the motion of a particle in a plane can be described by giving its Cartesian coordinates (x, y) as functions of time, together with the corresponding fixed orthogonal unit vectors $\mathbf{i}$ and $\mathbf{j}$.

In *Unit 28* you saw that it is sometimes more convenient to use polar coordinates $[r, \theta]$ and the corresponding orthogonal *unit polar vectors* $\mathbf{e}_r$ and $\mathbf{e}_\theta$ (see Figure 8). For instance, corresponding to Equation (5), the polar form of the position vector is

$$\mathbf{r} = r\mathbf{e}_r, \tag{8}$$

where $r = |\mathbf{r}|$ is the distance of the particle from the origin. This equation is not as simple as it looks because, for general motion in the plane, *both* r and $\mathbf{e}_r$ depend on time, as do the polar angle θ and its associated unit vector $\mathbf{e}_\theta$.

Unit 28 Subsection 4.2

The polar coordinates $[r, \theta]$ are defined in terms of the Cartesian coordinates (x, y) by the equations

$$x = r\cos\theta, \quad y = r\sin\theta.$$

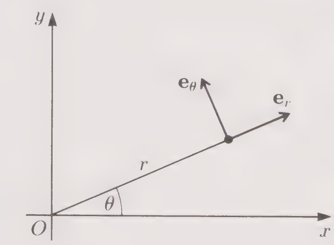


Figure 8

To see this time dependence, consider the fact that any vector in the plane can be expressed in terms of either the pair $\mathbf{i}, \mathbf{j}$ or the pair $\mathbf{e}_r, \mathbf{e}_\theta$. In particular, each of $\mathbf{i}$ and $\mathbf{j}$ themselves can be expressed as a linear combination of $\mathbf{e}_r, \mathbf{e}_\theta$, and vice versa. From *Unit 28*, or by noting that the unit vectors $\mathbf{e}_r, \mathbf{e}_\theta$ are obtained by rotating $\mathbf{i}, \mathbf{j}$ respectively through an angle θ (see Figure 9), these relations are

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \qquad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} \tag{9}$$

and

$$\mathbf{i} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta, \qquad \mathbf{j} = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta. \tag{10}$$

Equations (9) and (10) apply at all points of the plane other than the origin, where $\theta, \mathbf{e}_r$ and $\mathbf{e}_\theta$ are undefined.

Equations (9) show that $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are functions of the polar angle θ. Thus, if a particle is moving along some curve so that θ is time-dependent, then $\mathbf{e}_r$ and $\mathbf{e}_\theta$ depend on time too. For instance, Figure 10 shows the curve followed by a particle moving anticlockwise around and away from the origin, which passes through the point $[r_1, \theta_1]$ at time t_1 and the point $[r_2, \theta_2]$ at some later time t_2. Note that, while $\mathbf{e}_r$ and $\mathbf{e}_\theta$ depend upon θ, they are independent of the radial distance r (which does not appear in Equations (9)).

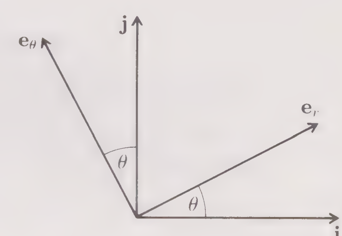


Figure 9

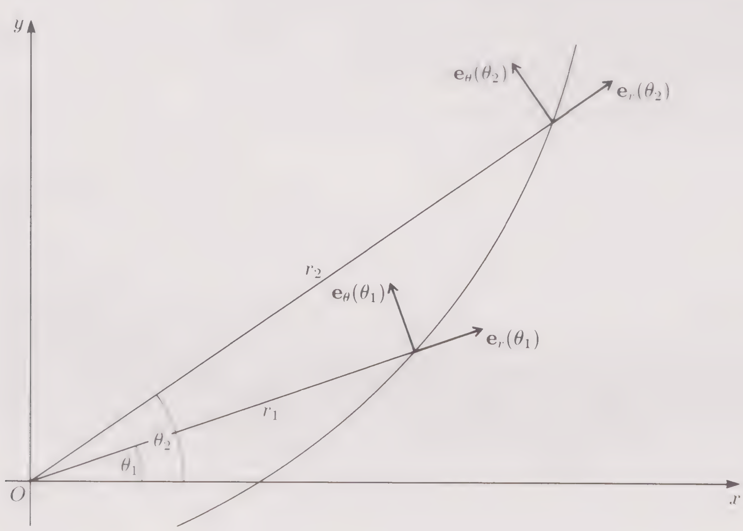


Figure 10

Exercise 9

Show explicitly from Equations (9) that, at each point other than the origin, $\mathbf{e}_r, \mathbf{e}_\theta$ form an orthogonal pair of unit vectors, that is, they satisfy the conditions

$$\mathbf{e}_r \cdot \mathbf{e}_\theta = 0 \quad \text{and} \quad \mathbf{e}_r \cdot \mathbf{e}_r = \mathbf{e}_\theta \cdot \mathbf{e}_\theta = 1.$$

(You will need to use the fact that $\mathbf{i}, \mathbf{j}$ form an orthogonal pair of unit vectors.)

[*Solution on page 48*]

In order to study the motion of a particle in plane polar coordinates, it is necessary to have expressions for its velocity and acceleration vectors in terms of $r, \theta, \mathbf{e}_r$ and $\mathbf{e}_\theta$. In the course of deriving these equations, we need to express the time derivatives $\dot{\mathbf{e}}_r, \dot{\mathbf{e}}_\theta$ in terms of $\mathbf{e}_r, \mathbf{e}_\theta$. The required relations for achieving this are

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \quad \text{and} \quad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r, \tag{11}$$

which may be obtained by differentiating Equations (9) and applying the chain rule.

These equations were derived in *Unit 28* Subsection 4.2. Although attention was restricted there to circular motion, the derivations apply also in the general case.

We shall now apply Equations (11) to obtain expressions for the velocity $\mathbf{v} = \dot{\mathbf{r}}$ and acceleration $\mathbf{a} = \ddot{\mathbf{r}}$ in terms of polar coordinates and unit vectors. As pointed out above, the position vector $\mathbf{r}$ is given by

$$\mathbf{r} = r\mathbf{e}_r, \tag{8}$$

where $r = |\mathbf{r}|$. Differentiating this equation and using the product rule, we have

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r,$$

which by the first of Equations (11) may be written as

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta. \tag{12}$$

Note that $\dot{r} = d|\mathbf{r}|/dt$, the rate of change of the radial coordinate, is *not* in general the same as the speed $|\dot{\mathbf{r}}|$ of the particle.

Differentiation of this expression for the velocity produces the relation

$$\ddot{\mathbf{r}} = \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta$$

for the acceleration. After applying each of Equations (11) and collecting terms, we arrive at

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta.$$

These expressions for the velocity and the acceleration differ from those derived in *Unit 28* Subsection 4.2 because there we restricted ourselves to circular motion with $r =$ constant.

Recognizing that

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}),$$

we can write the expression for the acceleration as

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\mathbf{e}_\theta. \tag{13}$$

The following box summarizes the results which we have obtained.

Position, velocity and acceleration in plane polar coordinates

The plane polar unit vectors $\mathbf{e}_r, \mathbf{e}_\theta$ are related to the Cartesian unit vectors $\mathbf{i}, \mathbf{j}$ by the equations

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \qquad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} \tag{9}$$

and

$$\mathbf{i} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta, \qquad \mathbf{j} = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta. \tag{10}$$

The derivatives $\dot{\mathbf{e}}_r, \dot{\mathbf{e}}_\theta$ are given in terms of $\mathbf{e}_r, \mathbf{e}_\theta$ by

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \quad \text{and} \quad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r. \tag{11}$$

In terms of r, θ, $\mathbf{e}_r$ and $\mathbf{e}_\theta$, the position $\mathbf{r}$, velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ of a particle moving in the (x, y)-plane are given by the equations

$$\mathbf{r} = r\mathbf{e}_r, \tag{8}$$

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta, \tag{12}$$

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\mathbf{e}_\theta. \tag{13}$$

Of the following four exercises, the first three will give you practice in expressing a particle's velocity and acceleration in plane polar coordinates. The fourth, Exercise 13, asks you to express the mechanical energy of a particle in polar coordinates. The result of this exercise will be applied later in the unit.

Exercise 10

A particle is constrained to move in a circle of fixed radius $r = R$ whose centre is at the origin. Find the velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ in terms of R, $\dot{\theta}$, $\ddot{\theta}$, $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Give the velocity and acceleration also in terms of R, θ, $\dot{\theta}$, $\ddot{\theta}$, $\mathbf{i}$ and $\mathbf{j}$.

This is the particular type of planar motion which was considered in *Unit 28* Section 3 and Subsection 4.2.

Exercise 11

A particle is constrained to move radially outwards from the origin, at some fixed angle $\theta = \theta_0$. Find the particle's velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ in terms of $\dot{r}$, $\ddot{r}$ and $\mathbf{e}_r$. Give the velocity and acceleration also in terms of $\dot{r}$, $\ddot{r}$, θ_0, $\mathbf{i}$ and $\mathbf{j}$.

Exercise 12

A particle moves in the (x, y)-plane with radial coordinate $r(t) = at^2 + bt$ and angle $\theta(t) = ct^2$. Find the velocity and acceleration in terms of t, $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

Exercise 13

It was shown in Subsection 1.2 that if a particle moves under the influence of a conservative force $\mathbf{F} = -\operatorname{grad} U$, then its total mechanical energy $E = \frac{1}{2}m\dot{\mathbf{r}}^2 + U$ is constant. Show that if the motion takes place in the (x, y)-plane then E may be written as

Recall that by $\dot{\mathbf{r}}^2$ we mean $\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = |\dot{\mathbf{r}}|^2$.

$$E = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U.$$

[*Solutions on page 48*]

Thus far we have derived kinematical formulas for the planar motion of a particle, using polar coordinates. Before proceeding to the subject of planetary orbits we shall briefly examine the mechanics involved in some simpler systems. The starting point for this examination is, as usual, Newton's second law.

In plane Cartesian coordinates, Newton's second law $m\ddot{\mathbf{r}} = \mathbf{F}$ takes the form

$$m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}) = F_x\mathbf{i} + F_y\mathbf{j},$$

Here F_x and F_y are respectively the x- and y-components of the force $\mathbf{F}$.

leading as in *Unit 15* Section 2 to the two scalar equations

$$m\ddot{x} = F_x \quad \text{and} \quad m\ddot{y} = F_y.$$

Similarly, Newton's second law $m\ddot{\mathbf{r}} = \mathbf{F}$ may be expressed in plane polar coordinates, using Equation (13), as

$$m\left((\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\mathbf{e}_\theta\right) = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta,$$

Here F_r and F_θ are respectively the *radial* and *transverse* components of the force $\mathbf{F}$.

giving the pair of scalar equations

$$m(\ddot{r} - r\dot{\theta}^2) = F_r \quad \text{and} \quad \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = F_\theta. \tag{14}$$

It is these equations of motion which are sought in the following example and exercises, each of which represents a situation in which it is appropriate to use polar rather than Cartesian coordinates.

Example 2

A particle of mass m is launched at the bottom of a circular track of radius R (see Figure 11). As the particle moves along the track it is subjected to the downward force of gravity, a normal reaction force of magnitude F perpendicular to the track, and a frictional force of magnitude f in the direction opposite to that of the particle's motion.

Set up the equations of motion for the particle in plane polar coordinates, assuming that the particle remains in contact with the track and moves in the sense of increasing θ.

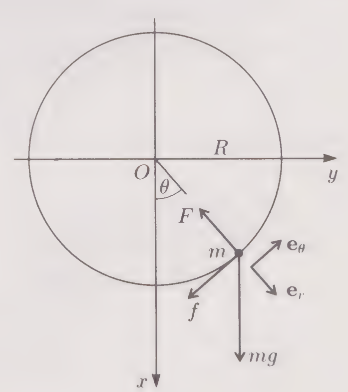


Figure 11

Solution

The forces have been drawn on Figure 11, which also indicates our choice of Cartesian coordinate system. The reaction exerted by the track on the particle is $-F\mathbf{e}_r$. The force of gravity is $mg\mathbf{i}$, but $\mathbf{i}$ is given in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ by the first of Equations (10), so the gravitational force may be written as

$$mg(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta).$$

The particle moves in the sense of increasing θ, that is, in the direction of $\mathbf{e}_\theta$, so the frictional force is $-f\mathbf{e}_\theta$. Adding the three forces together gives the total force

$$-F\mathbf{e}_r + mg(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) - f\mathbf{e}_\theta = (-F + mg\cos\theta)\mathbf{e}_r + (-f - mg\sin\theta)\mathbf{e}_\theta,$$

so that the radial and transverse components of the total force are respectively

$$F_r = -F + mg\cos\theta \quad \text{and} \quad F_\theta = -f - mg\sin\theta.$$

Since the particle remains in contact with the track, it is constrained to move in a circle with constant radial coordinate $r = R$, so that $\dot{r} = \ddot{r} = 0$. The equations of motion (Equations (14)) are therefore

$$-mR\dot{\theta}^2 = -F + mg\cos\theta,$$
$$mR\ddot{\theta} = -f - mg\sin\theta. \quad \square$$

With $f = 0$, these are also the equations of motion for a circular pendulum, where F is now the tension in the string. You saw these equations derived via Cartesian coordinates in *Unit 15* Section 5, and more directly in *Unit 28* Subsection 3.3.

Exercise 14

A puck of mass m moves without friction around the origin O on a horizontal table (see Figure 12). It is fixed to a light, inextensible string which is reeled in steadily, so that its distance from the origin is given by

$$r(t) = r(0) - \mu t \qquad (t < r(0)/\mu),$$

where μ is a positive constant. The string remains taut throughout.

The upper bound on the time t is included because $r(t)$ must always be non-negative. The puck would reach the origin at time $t = r(0)/\mu$ if the situation remained as described until that time.

(i) Show that the equations of motion for the particle which represents the puck are

$$mr\dot{\theta}^2 = T \quad \text{and} \quad \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0,$$

where T is the tension in the string.

(ii) Show that at time t the tension is given by

$$T(t) = \frac{mr^4(0)\dot{\theta}^2(0)}{r^3(t)} = \frac{mr^4(0)\dot{\theta}^2(0)}{(r(0) - \mu t)^3}.$$

(iii) Suppose that $m = 0.1\,\text{kg}$, $\mu = \frac{1}{60}\,\text{m}\,\text{s}^{-1}$, the initial distance is $r(0) = 1\,\text{m}$ and the initial angular velocity is $\dot{\theta}(0) = 1\,\text{rad}\,\text{s}^{-1}$. Suppose further that the string will break when it experiences a tension of 500 N. Find the distance r and the time t at which the string breaks.

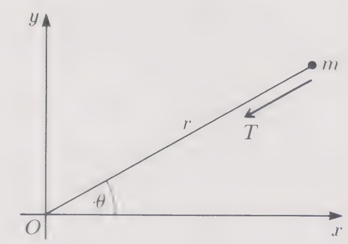


Figure 12

Exercise 15

A puck of mass m moves frictionlessly around the origin on a horizontal table. It is connected to the origin by a perfect spring of natural length l_0 and stiffness k, so that it is acted upon by a force

$$\mathbf{F} = -k(r - l_0)\mathbf{e}_r.$$

You will see this apparatus demonstrated and discussed further in the television programme for this unit (Section 4).

(i) Set up the equations of motion for the particle which represents the puck.

(ii) Show that, however the particle moves, $mr^2\dot{\theta}$ is constant.

(iii) Verify that a potential energy function for this force is $U = \frac{1}{2}k(r - l_0)^2$.

(iv) Suppose that the puck has speed u when the spring has length $2l_0$. What is its speed when the spring has its natural length in the subsequent motion?

[*Solutions on page 48*]

Summary of Section 1

1. For particle motion in three or fewer dimensions, a force $\mathbf{F}$ acting on the particle is **conservative** if there exists a scalar field function $U = U(\mathbf{r}) = U(x, y, z)$ such that

$$\mathbf{F} = -\operatorname{grad} U.$$

The function U is the **potential energy** of the particle. In such a case, the total mechanical energy of the particle,

$$E = \tfrac{1}{2}m\dot{\mathbf{r}}^2 + U$$

(where $\dot{\mathbf{r}}^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = |\dot{\mathbf{r}}|^2$), is constant throughout the motion.

2. In one dimension, the condition for a conservative force and the expression for the total mechanical energy become respectively

$$F = -\frac{dU}{dx} \quad \text{and} \quad E = \tfrac{1}{2}m\dot{x}^2 + U,$$

where F is the x-component of the force and $U = U(x)$.

3. The unit vectors $\mathbf{e}_r, \mathbf{e}_\theta$ corresponding to the plane polar coordinates $[r, \theta]$ (where $x = r\cos\theta$, $y = r\sin\theta$) are related to the Cartesian unit vectors $\mathbf{i}, \mathbf{j}$ by the equations

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \qquad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

and $$\mathbf{i} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta, \qquad \mathbf{j} = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta.$$

4. The derivatives $\dot{\mathbf{e}}_r, \dot{\mathbf{e}}_\theta$ are given in terms of $\mathbf{e}_r, \mathbf{e}_\theta$ by

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \quad \text{and} \quad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r.$$

5. The position $\mathbf{r}$, velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ of a particle moving in the (x, y)-plane are given in terms of plane polar coordinates by the equations

$$\mathbf{r} = r\mathbf{e}_r,$$
$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta,$$
$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\mathbf{e}_\theta.$$

2 Kepler's laws and Newton's law of gravitation

The Earth is an ordinary-sized planet which orbits a middle-aged star of rather small size near the outer fringes of a typical spiral galaxy some 10^5 light-years across. This galaxy contains between 10^{11} and 10^{12} visible stars, together with much other matter, and is one member of a rough association, or cluster, of some thousand galaxies of various shapes and sizes.

One light-year is the distance travelled by light in one year, or approximately 9.5×10^{15} m.

Located in this myriad of astronomical objects is our solar system. It consists of a relatively massive Sun encircled by eight chief planets, some of which themselves possess one or more satellite moons. Any member of the solar system interacts via gravitational force with all of the others and, indeed, with all other heavenly bodies. But distances between stars are measured in light-years—and the force of gravity between two bodies decreases rapidly in magnitude as their separation increases—so interactions with objects beyond the solar system can mostly be neglected.

Owing to its small size and features of its orbit, Pluto is nowadays not considered to be a planet.

The gravitational influence of a body increases with mass, and within the solar system, there is a rough hierarchy of size. The Sun is by far the largest object, followed by the planets and then by their moons. The Earth's mass is only about 3×10^{-6} that of the Sun, and its moon's mass is about 10^{-2} of its own. The largest planet, Jupiter, is roughly 300 times more massive than the Earth, but it is far away. In fact, the gravitational influence of a planet upon any other body is of little significance compared with the effect of the Sun, unless the body acted upon is relatively close to the planet. To a good first approximation, then, we can think of the planets as orbiting the Sun singly, and the moons as singly orbiting their planets. This is the model adopted by Newton, who considered the problem of only *two* bodies orbiting one another. (In a more advanced treatment, the effects of the other planets can be added as perturbations.) We shall assume in addition that one of the bodies is very much more massive than the other.

An analysis of the motion of one relatively light body moving under the gravitational attraction of a second, relatively massive, body is the goal of Section 3. In this section we introduce the topic of planetary motion by describing (in Subsection 2.1) the features of their orbits which were observed by Kepler and stated in the form of three informative laws. These laws will be derived in Section 3 as a consequence of Newton's *universal law of gravitation*, which we consider in Subsection 2.2.

2.1 Kepler's laws described

As they are worded, Kepler's three laws refer specifically to the orbits of planets around the Sun, but they apply also to the orbit of any relatively light body about another much more massive body. The laws were established by Kepler from experimental observations of the motion of several of the planets. You will see later that they may be derived from Newton's three laws together with his law of gravitation. It is the purpose of the current subsection to explain the meaning of Kepler's laws.

> **Kepler's laws of planetary motion**
>
> **Law I** Each planet moves in an ellipse, with the Sun at one focus.
>
> **Law II** The line joining a planet to the Sun sweeps out equal areas in equal times.
>
> **Law III** The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

Kepler's first law specifies the *shape* of a planetary orbit. Since it refers to an *ellipse* and to a *focus*, we start by considering the mathematics of ellipses.

An ellipse is a symmetrical plane figure like that drawn in Figure 1. One of the implications of this first law, then, is that the orbit of a planet about the Sun *lies in a plane*. The usual definition of an **ellipse** is as a set of points (X, Y) which, with a suitable choice of origin and axes, satisfy the equation

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1, \qquad (1)$$

where a and b are positive constants with $a \geq b$. The constants a and b are called respectively the *semi-major axis* and *semi-minor axis* of the ellipse. Their geometrical significance is shown in Figure 1.

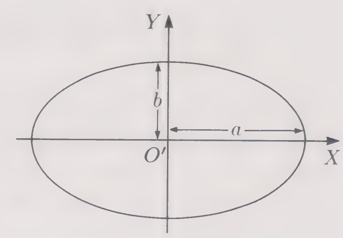


Figure 1

Exercise 1

What can be said about an ellipse in the special case $a = b$?

Exercise 2

(i) Show that the area A of an ellipse can be expressed as

$$A = 2b \int_{-a}^{a} \sqrt{1 - (X/a)^2}\, dX.$$

(ii) By using the substitution $X = a \cos u$, show that $A = \pi ab$.

[*Solutions on page 49*]

To understand Kepler's first law, it remains to explain where a focus of an ellipse is located, and then to express the equation of the ellipse relative to this point. We shall express the equation with respect to the new set of axes Oxy shown in Figure 2 below. These axes are obtained from the set $O'XY$ of Figure 1 by a translation of the axes to the right by a distance q, where

$$q = \sqrt{a^2 - b^2}. \qquad (2)$$

(We could also shift axes to the left by the distance q, but our choice represents no loss of generality since this configuration is just the other rotated by an angle π.) The points F and O of Figure 2, located a distance q to the left and to the right of the centre of the ellipse, are called the *foci* of the ellipse.

'Foci' is the plural of 'focus'.

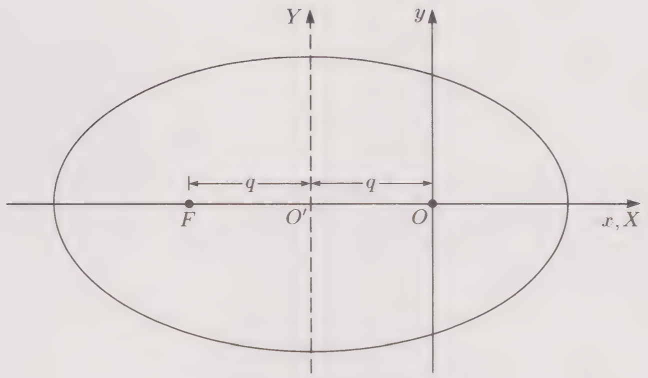


Figure 2

Exercise 3

Where are the foci of a circle?

[*Solution on page 49*]

Since the translation is along the coincident x- and X-axes, the new coordinates (x, y) are related to the original ones (X, Y) by

$$X = x + q, \qquad Y = y.$$

Then, from Equation (1), the equation of the ellipse takes the form

$$\frac{(x+q)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We shall now proceed to re-express this equation in plane polar coordinates. After multiplying the equation by b^2 and rearranging, we obtain

$$y^2 = b^2 - \frac{b^2}{a^2}(x+q)^2 = b^2 - \frac{b^2}{a^2}(x^2 + 2qx + q^2).$$

Adding x^2 to both sides and collecting terms gives

$$x^2 + y^2 = \left(b^2 - \frac{b^2}{a^2}q^2\right) - \frac{2b^2 q}{a^2}x + \left(1 - \frac{b^2}{a^2}\right)x^2.$$

The left-hand side is r^2, the square of the radial distance, so that, on using Equation (2) to express q in terms of a, b, we have

$$r^2 = \frac{b^4}{a^2} - 2\frac{b^2 x}{a}\sqrt{1 - \frac{b^2}{a^2}} + \left(1 - \frac{b^2}{a^2}\right)x^2. \qquad (3)$$

This rather cumbersome expression is made neater by employing two new quantities defined in terms of a and b. These quantities are the *eccentricity*

This usage of the symbol e should not be confused with its use as the base of natural logarithms, 2.718 28. . . .

$$e = \frac{q}{a} = \sqrt{1 - \frac{b^2}{a^2}} \qquad (4)$$

and the *semi-latus rectum*

$$l = \frac{b^2}{a}. \qquad (5)$$

Note that, since $b \leq a$, we have $0 \leq e \leq 1$. This is a condition on the eccentricity for ellipses.

Exercise 4

Find e when

(i) $a = b$ (a circle); (ii) $b = \frac{1}{2}a$; (iii) $b = \frac{1}{3}a$.

Comment on the relationship between the eccentricity and shape of an ellipse.

[*Solution on page 49*]

From Equations (3)–(5), we now have

$$r^2 = l^2 - 2lex + e^2x^2$$

or $$r^2 = (l - ex)^2.$$

Then taking the square root of both sides gives

$$r = \pm(l - ex). \qquad (6)$$

Putting $x = 0$ into this equation gives $r = \pm l$, showing that the semi-latus rectum l is the distance from the origin of the ellipse's intercepts on the y-axis. This is illustrated in Figure 3 below.

We must now decide on the choice of sign for the right-hand side of Equation (6), to be consistent with the fact that r is never negative. We shall show that, for all points x on the ellipse, the quantity $l - ex$ is positive. From Figure 2, the largest value of x on the ellipse is $x = a - q$. Thus we have, for all points on the curve,

$$l - ex \geq l - e(a - q).$$

But we also have $q = \sqrt{a^2 - b^2}$, $l = b^2/a$ and $e = q/a$, so the right-hand side of this inequality is

$$\begin{aligned} l - e(a - q) &= \frac{b^2}{a} - \frac{\sqrt{a^2 - b^2}}{a}\left(a - \sqrt{a^2 - b^2}\right) \\ &= \frac{b^2}{a} - \sqrt{a^2 - b^2} + \frac{a^2 - b^2}{a} \\ &= a - \sqrt{a^2 - b^2}, \end{aligned}$$

which is never negative. Hence $l - ex$ itself is never negative, when x assumes values for points on the curve. Thus the positive sign is appropriate in Equation (6), and the equation for an ellipse with respect to the axes Oxy of Figure 2 is

$$r = l - ex \quad \text{or} \quad \frac{l}{r} = 1 + e\frac{x}{r}.$$

Since $x = r\cos\theta$ (see Figure 3) we have the convenient form

$$\frac{l}{r} = 1 + e\cos\theta \qquad (7)$$

for the equation of an ellipse in polar coordinates with origin at a focus of the ellipse.

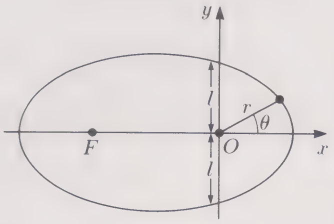


Figure 3

Kepler's first law therefore says that if the origin of coordinates is located at the Sun, then the orbit of each planet lies in a plane containing that origin, and is described by Equation (7) once the axes have been appropriately oriented. With a different orientation for the axes (but the same origin), Equation (7) becomes

$$\frac{l}{r} = 1 + e\cos(\theta - \theta_0), \qquad (8)$$

where θ_0 is the clockwise angle through which the axes have been rotated from their position in Figure 3. The major axis of the ellipse then lies along the direction specified by $\theta = \theta_0$.

Kepler's second law says that the line joining a planet to the Sun sweeps out equal areas in equal times. This law can be used to specify the *rate* at which a planet moves in its particular elliptical orbit.

Suppose, without loss of generality, that the planet moves in an anticlockwise sense. In Figure 4 the two shaded regions represent the areas swept out by the planet between times t_1 and t_2, and between times t_3 and t_4. Assuming that the two time intervals are of equal duration, Kepler's second law says that these areas are equal. If $A(t)$ is the total area swept out since some initial time, $t = 0$ say, this law can be expressed as

$$A(t_4) - A(t_3) = A(t_2) - A(t_1) \quad \text{whenever} \quad t_4 - t_3 = t_2 - t_1.$$

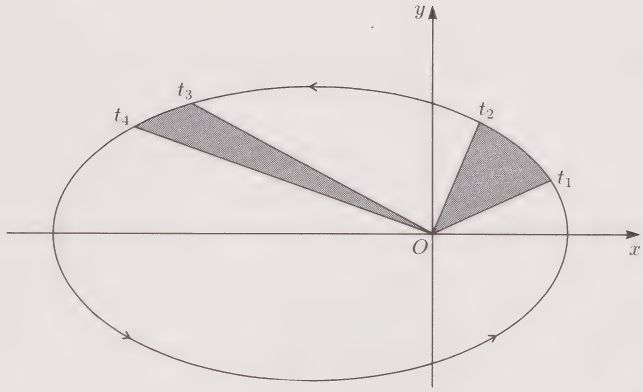


Figure 4

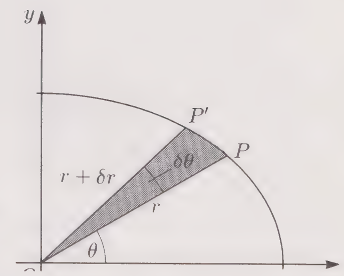


Figure 5

If $r(t)$ is decreasing then the expressions for the upper and lower bounds in this inequality are reversed, leading to the same outcome.

A more concise way of stating Kepler's second law is that the *rate* at which area is swept out is constant, that is,

$$\dot{A} \text{ is constant.}$$

It is possible also to state the law in a form which relates the radial coordinate r to the rate of change $\dot{\theta}$ of the angular coordinate. Figure 5 shows a close-up of part of the planetary orbit, where we consider the area δA swept out during a small time interval of duration δt. Over this interval the planet moves from point P at time t to point P' at time $t + \delta t$, traversing an angle $\delta\theta$ and changing its radius from r to $r + \delta r$. If δt is sufficiently small then the area δA will lie between the areas of two circular sectors with angle $\delta\theta$, one having radius r and the other radius $r + \delta r$. If $r(t)$ is increasing in the part of the curve under consideration then this gives

$$\tfrac{1}{2}r^2\delta\theta \leq \delta A \leq \tfrac{1}{2}(r + \delta r)^2\delta\theta.$$

On dividing through by δt, and then taking the limit as δt tends to zero, we obtain

$$\dot{A} = \tfrac{1}{2}r^2\dot{\theta}. \qquad (9)$$

Since $\dot{A}$ is constant according to Kepler's second law, this law may also be interpreted as saying that

$$r^2\dot{\theta} \text{ is constant.}$$

Equation (9) can also be used to express an area swept out by the planet in terms of r and $\dot{\theta}$. Thus the shaded area in Figure 4 covered between times t_1 and t_2 is

$$A(t_2) - A(t_1) = [A]_{t_1}^{t_2} = \int_{t_1}^{t_2} \dot{A}\,dt = \int_{t_1}^{t_2} \tfrac{1}{2}r^2(t)\dot{\theta}(t)\,dt.$$

Exercise 5

A body moves in a circular orbit around the origin, with radius R and fixed speed V. Show that $\dot{A}$ is constant, and find its value.

Exercise 6

Compare the average speeds of the orbiting planet during the two equal intervals $t_1 \leq t \leq t_2$ and $t_3 \leq t \leq t_4$ in Figure 4. Your answer should consist of a few sentences only.

[*Solutions on page 49*]

Kepler's third law gives a relation between the period of the planetary orbit and the lateral extent (semi-major axis) of the orbit. If T is the period then, by definition, the planet will take this time to make one full circuit of the ellipse. If the particular orbit has semi-major axis a, then Kepler's third law says that T^2 is proportional to a^3, that is,

$$T^2 = ka^3,$$

where k is a constant. This law, then, makes a statement connecting the overall rate of motion of the planet to a geometrical property of its orbit. We shall prove this result, together with Kepler's other two laws, in Section 3.

Exercise 7

The semi-major axis of the planet Mars is 1.524 times as large as the Earth's semi-major axis. The Earth's period for its orbit around the Sun is 365.256 days. Find the period (in Earth days) for the orbit of Mars around the Sun.

[*Solution on page 49*]

2.2 Gravitational force and potential energy

> I began to think of gravity extending to ye orb of the Moon, and ... from Kepler's Rule ... I deduced that the forces which keep the Planets in their Orbs must [vary] reciprocally as the square of their distances from the centres about which they revolve: and thereby compared the force requisite to keep the Moon in her Orb with the force of gravity at the surface of the Earth, and found them to answer pretty nearly. All this was in the two plague years of 1665 and 1666, for in those days I was in the prime of my age for invention, and minded Mathematicks and Philosophy more than at any time since.
>
> *Isaac Newton's account of his discovery of the universal law of gravitation.*

This subsection starts with the statement of Newton's universal law of gravitation, then derives the potential energy function corresponding to the (conservative) gravitational force exerted by a particular type of extended object. Finally, we compare the theory of gravitation introduced here with the simplified version which has been applied in previous units for motion near the surface of the Earth.

You met Newton's law of gravitation in the television programme for *Unit 28*.

Newton's *universal law of gravitation* describes the nature of the force between pairs of particles. It gives the attractive force between them as a function of the masses of the particles and of their separation. It is *universal* in the sense that it is hypothesized to hold between the particles comprising all bodies in the universe, and not just between those comprising the Earth and an apple, or the Earth and the Moon. In words, the law can be stated as follows.

Newton's universal law of gravitation

The force of gravity between two particles is attractive, directed along the line between them, proportional to the product of their masses, and inversely proportional to the square of their separation.

Let us write this mathematically, using the vector notation which was unavailable to Newton. We write the position vectors of the two particles as $\mathbf{r}_i, \mathbf{r}_j$ and their masses as m_i, m_j. The gravitational force on m_i due to the presence of m_j is denoted by $\mathbf{F}_{ij}$. From the verbal statement of Newton's law above, we know the following.

As a shorthand device, we shall refer to the particles by using their masses m_i, m_j as labels.

(i) $\mathbf{F}_{ij}$ is directed from m_i to m_j, hence its direction is defined by the unit vector $\dfrac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|}$ (see Figure 6).

(ii) The magnitude of $\mathbf{F}_{ij}$ is proportional to $m_i m_j$, and to $\dfrac{1}{|\mathbf{r}_j - \mathbf{r}_i|^2}$.

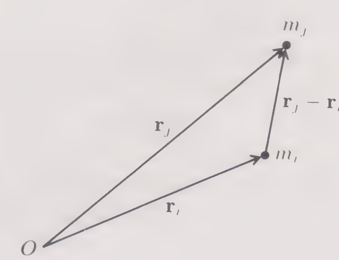


Figure 6

Putting these elements together gives the force of gravity on m_i exerted by m_j as

$$\mathbf{F}_{ij} = \frac{Gm_i m_j}{|\mathbf{r}_j - \mathbf{r}_i|^3}(\mathbf{r}_j - \mathbf{r}_i). \qquad (10)$$

This is **Newton's universal law of gravitation** in vector form. Here G is the constant of proportionality, called the universal constant of gravitation, or *gravitational constant*. It is one of the least precisely measured physical constants. Its value, in SI units, is

$$G \simeq 6.67 \times 10^{-11}. \qquad (11)$$

Exercise 8

What are the SI units of G?

Exercise 9

Show that $\dfrac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|}$ is a unit vector.

Exercise 10

Gravitation is an inter-particle force. By Newton's third law (*Unit 17* Section 2), the force exerted upon m_i by m_j should be equal in magnitude, but opposite in direction, to the force exerted upon m_j by m_i. Show that the universal law of gravitation is consistent with this, so that $\mathbf{F}_{ij} + \mathbf{F}_{ji} = \mathbf{0}$.

[*Solutions on page 49*]

Most of the rest of this subsection is taken up with consideration of the gravitational attraction caused by an extended object with *spherically symmetric* mass distribution, and calculation of the corresponding potential energy function.

If you are short of time, you might like to read as far as the paragraph below Figure 7 and then move directly to the summary box on page 25.

Using Equations (10) and (11), it is possible to calculate the force of attraction between any two objects, provided that they are sufficiently small to be modelled by particles. In many cases, however, it is far from the truth to claim that both objects are 'small' relative to the situation being considered. For example, Figure 7(a) shows a satellite of mass m, in orbit around the Earth at a height approximately equal to the Earth's own radius. At first sight it seems unreasonable here to model the Earth by a particle, although it appears sensible (for most purposes) to use a particle model for the satellite.

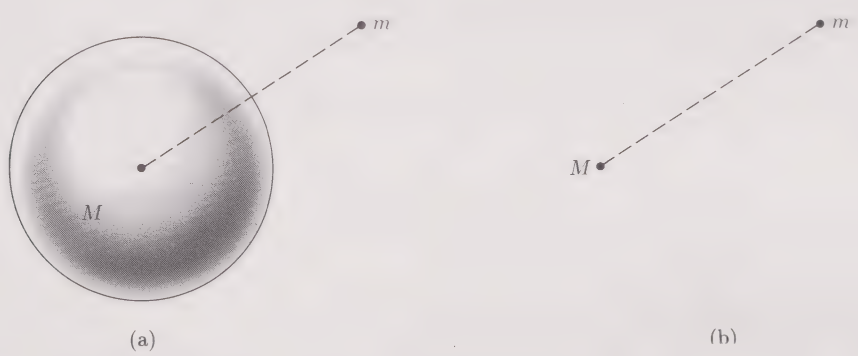


Figure 7

In fact, you will see in this subsection that if the Earth's mass is assumed to be distributed in a *spherically symmetric* manner (to be defined below), then its attraction on the satellite is the *same* as if all its mass were concentrated in a particle at its centre (see Figure 7(b)).

To approach this problem, consider the gravitational force on some particle of mass m outside an arbitrary extended object. We want to calculate the net force acting on the particle with respect to some coordinate system $Oxyz$, as in Figure 8. To do this, we regard the extended body as being composed of many small volume elements. Then the net force on the exterior particle is obtained by adding together all the (vector) contributions from these volume elements. Since the elements are small, they can be treated individually as particles.

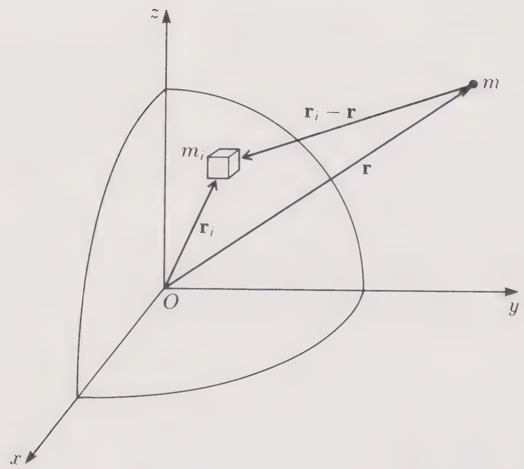


Figure 8

The force exerted upon the particle of mass m by a typical volume element, which has mass m_i and position vector $\mathbf{r}_i$, is obtained by applying Equation (10). This force is

$$Gmm_i\frac{\mathbf{r}_i - \mathbf{r}}{|\mathbf{r}_i - \mathbf{r}|^3},$$

where $\mathbf{r}$ is the position vector of the mass m. Adding together the contributions from all the volume elements in the extended object gives the net gravitational force on m as

$$\mathbf{F}(\mathbf{r}) = Gm\sum_{i=1}^{N}\frac{m_i(\mathbf{r}_i - \mathbf{r})}{|\mathbf{r}_i - \mathbf{r}|^3}. \qquad (12)$$

This is the gravitational force on the particle of mass m due to the system of N particles. Although this is only an approximation to the gravitational effect of the extended object, the approximation becomes exact in the limit as N is increased indefinitely.

By construction, the sum in Equation (12) supposes that the extended object consists of a large number, N, of small elements m_i. We shall ultimately consider (for a particular case) the limit of this sum as N becomes large and each m_i tends to zero, which leads to an expression for $\mathbf{F}(\mathbf{r})$ as an integral over the volume of the extended object. This could be done by taking the limit directly from Equation (12), but it is simpler to rewrite it first in terms of gravitational *potential energy*.

You saw in Subsection 1.2 that a force $\mathbf{F}(\mathbf{r})$ is conservative, with potential energy function $U(\mathbf{r})$, provided that

$$\mathbf{F} = -\operatorname{grad} U.$$

Equation (3) of Section 1 (page 9)

We shall now show that the net force given by Equation (12) can be expressed in this way. The main step in this demonstration is to establish that

$$\frac{\mathbf{r}_i - \mathbf{r}}{|\mathbf{r} - \mathbf{r}_i|^3} = \operatorname{grad}\frac{1}{|\mathbf{r} - \mathbf{r}_i|}. \qquad (13)$$

In order to prove Equation (13), it suffices to use Cartesian coordinates, in terms of which we have

$$|\mathbf{r} - \mathbf{r}_i| = \left((x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2\right)^{1/2}.$$

The definition of the gradient gives

$$\operatorname{grad}\frac{1}{|\mathbf{r} - \mathbf{r}_i|} = \mathbf{i}\frac{\partial}{\partial x}\frac{1}{|\mathbf{r} - \mathbf{r}_i|} + \mathbf{j}\frac{\partial}{\partial y}\frac{1}{|\mathbf{r} - \mathbf{r}_i|} + \mathbf{k}\frac{\partial}{\partial z}\frac{1}{|\mathbf{r} - \mathbf{r}_i|}. \qquad (14)$$

The $\mathbf{i}$-component on the right-hand side of this equation is

$$\begin{aligned}\mathbf{i}\frac{\partial}{\partial x}\frac{1}{|\mathbf{r} - \mathbf{r}_i|} &= \mathbf{i}\frac{\partial}{\partial x}\left((x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2\right)^{-1/2} \\ &= \mathbf{i}(-\tfrac{1}{2})\left((x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2\right)^{-3/2} \times 2(x - x_i) \\ &= -\mathbf{i}\frac{x - x_i}{|\mathbf{r} - \mathbf{r}_i|^3}.\end{aligned}$$

The calculations for the $\mathbf{j}$- and $\mathbf{k}$-components of Equation (14) are similar. Using the relations

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \text{and} \quad \mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k},$$

we obtain

$$\operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}_i|} = -\left(\mathbf{i}(x-x_i)+\mathbf{j}(y-y_i)+\mathbf{k}(z-z_i)\right)\frac{1}{|\mathbf{r}-\mathbf{r}_i|^3}$$
$$= \frac{\mathbf{r}_i-\mathbf{r}}{|\mathbf{r}-\mathbf{r}_i|^3},$$

which proves the validity of Equation (13).

Combining Equations (12) and (13) then gives the gravitational force $\mathbf{F}$ in terms of a potential energy function, as

$$\mathbf{F}(\mathbf{r}) = Gm\sum_{i=1}^{N} m_i \operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}_i|} = \operatorname{grad}\left(Gm\sum_{i=1}^{N}\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\right)$$

or $$\mathbf{F}(\mathbf{r}) = -\operatorname{grad}\left(-Gm\sum_{i=1}^{N}\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\right).$$

This equation expresses the net gravitational force exerted by a system of N particles, with masses m_i and corresponding position vectors $\mathbf{r}_i$ $(i = 1, 2, \ldots, N)$, upon an external particle of mass m and position vector $\mathbf{r}$. It has the form

$$\mathbf{F}(\mathbf{r}) = -\operatorname{grad} U(\mathbf{r}),$$

where the potential energy function U is given by

$$U(\mathbf{r}) = -Gm\sum_{i=1}^{N}\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}. \qquad (15)$$

Exercise 11

A particle of mass M is fixed at the origin. Find the corresponding gravitational potential energy function U for a second particle which has mass m and position vector $\mathbf{r}$. Find also the gravitational force exerted upon the second particle by the first.

[*Solution on page 49*]

We seek now to progress from the gravitational potential energy of a particle due to a system of particles to the corresponding potential energy for a *continuous* distribution of mass, as in Figure 8. To achieve this, we characterize the distribution of mass by its *mass density function*, $\rho(\mathbf{r})$. This function is defined so that any small volume element having volume δV_i and centred at position $\mathbf{r}_i$ contains a mass m_i, where

The mass density function ρ was introduced in *Unit 27* Section 2, where it was used to calculate the overall mass of extended bodies.

$$m_i \simeq \rho(\mathbf{r}_i)\delta V_i.$$

Inserting this expression into Equation (15) gives, for the gravitational potential energy U of a particle due to a body with continuous mass distribution,

$$U(\mathbf{r}) \simeq -Gm\sum_{i=1}^{N}\frac{\rho(\mathbf{r}_i)\delta V_i}{|\mathbf{r}-\mathbf{r}_i|}.$$

This approximation to U becomes exact in the limit as the size of the volume elements δV_i tends towards zero and their number, N, increases indefinitely. In this limit, we obtain

$$U(\mathbf{r}) = -Gm\int_B \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV. \qquad (16)$$

The right-hand side here is a volume integral with respect to the variable $\mathbf{r}'$, extending over the entire body B.

Equation (16) gives quite generally the gravitational potential energy of a particle of mass m located outside *any* continuous distribution of mass, specified by the mass density function ρ. We now specialize to the case of a mass m lying outside a *spherically symmetric* distribution of mass.

By definition, a mass distribution is **spherically symmetric** if the corresponding mass density function $\rho(\mathbf{r})$ depends only upon the radial distance, $r = \sqrt{x^2+y^2+z^2}$. We express this spherical dependence of the mass density function by writing it as

$$\rho(\mathbf{r}) = \rho(|\mathbf{r}|) = \rho(r).$$

Exercise 12

Which of the following mass density functions ρ correspond to spherically symmetric mass distributions? (A, B and R are positive constants.)

(a) $\rho(\mathbf{r}) = A\left(x^3 + y^3 + z^3\right)e^{-Br^2} \qquad (r \geq 0)$

(b) $\rho(\mathbf{r}) = A\left(x^2 + y^2\right)e^{-Br^2} \qquad (r \geq 0)$

(c) $\rho(\mathbf{r}) = A\left(x^2 + y^2 + z^2\right)^2 e^{-Br^2} \qquad (r \geq 0)$

(d) $\rho(\mathbf{r}) = \begin{cases} Ar^3 & (0 \leq r \leq R) \\ 0 & (r > R) \end{cases}$

(e) $\rho(\mathbf{r}) = \begin{cases} A(x^2 + y^2) & (0 \leq r \leq R) \\ 0 & (r > R) \end{cases}$

(f) $\rho(\mathbf{r}) = \begin{cases} A & (0 \leq r \leq R) \\ 0 & (r > R) \end{cases}$

[*Solution on page 49*]

The integral on the right-hand side of Equation (16) was considered in *Unit 27* for the case of a spherically symmetric mass distribution. It was proved there that when $\rho(\mathbf{r}') = \rho(|\mathbf{r}'|)$ then

Unit 27 Subsection 4.3

$$\int_B \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|}\, dV = \frac{M}{r}, \qquad (17)$$

where

$$M = \int_B \rho(\mathbf{r}')\, dV \qquad (18)$$

is the total mass of the sphere B. By Equations (16), (17) and (18), the gravitational potential energy function for a spherically symmetric mass distribution is

If the body is of finite extent and has a spherically symmetric mass distribution, then it is necessarily a sphere.

$$U(\mathbf{r}) = -\frac{GmM}{r}. \qquad (19)$$

Exercise 13

Show that the gravitational force exerted on a particle of mass m due to a spherically symmetric mass distribution of mass M with its centre at the origin is

$$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{r^2}\mathbf{e}_r,$$

where $\mathbf{e}_r = \mathbf{r}/|\mathbf{r}|$ is the unit vector in the outward radial direction.

[*Solution on page 49*]

Equation (19), together with the results of Exercises 11 and 13, establishes the truth of the statement following Figure 7: the gravitational attraction of a body with spherically symmetric mass distribution on an exterior particle is the same as if all of the body's mass were concentrated at its centre. In proving this result we have derived expressions for the gravitational force and potential energy which will be applied in Section 3 to the task of specifying planetary orbits. These expressions are given once again below.

The gravitational effect of a spherically symmetric body

A body has a *spherically symmetric* mass distribution if (with the origin of coordinates at the centre of the body) its mass density function $\rho(\mathbf{r})$ depends only on the radial distance $|\mathbf{r}| = r = \sqrt{x^2 + y^2 + z^2}$. The gravitational force exerted by such a body B on an exterior particle with mass m and position vector $\mathbf{r}$ is given by

$$\mathbf{F}(\mathbf{r}) = -\operatorname{grad} U(\mathbf{r}) = -\frac{GmM}{r^2}\mathbf{e}_r,$$

where $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector in the outward radial direction,

$$U(\mathbf{r}) = -\frac{GmM}{r} \tag{19}$$

is the gravitational potential energy of the particle, and

$$M = \int_B \rho(\mathbf{r}')\,dV \tag{18}$$

is the total mass of the body.

It is said in some quarters that it took Newton twenty years to prove this result, causing him to delay publishing *Principia*. But this version of events has not been established to the satisfaction of all science historians.

Exercise 14

Two spherically symmetric bodies have masses M and m. If their centres are separated at some instant by a distance R, what are then the magnitude and direction of the gravitational forces exerted by each body upon the other? [Hint: By a result of *Unit 17* Section 2, the motion of the centre of mass of a body is the same as that of a particle of the same total mass which experiences all of the external forces applied to particles of the body.]

[*Solution on page 50*]

To conclude this section, we examine how Newton's universal law of gravitation matches up with the simpler model used previously for motion near the surface of the Earth. When modelling motion close to the Earth it is customary to assume, as we have done in earlier units, that the force on a particle of mass m is directed vertically downwards and has the constant magnitude mg, where $g = 9.81\,\mathrm{m\,s^{-2}}$. More generally, under the assumption that the Earth is a spherically symmetric distribution of mass, we have just shown that at any point on or above the Earth's surface the gravitational force is

$$\mathbf{F}(\mathbf{r}) = -\frac{mMG}{r^2}\mathbf{e}_r,$$

where M is the mass of the Earth, $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector in the outward radial direction and $r = |\mathbf{r}|$ is the distance from the centre.

Clearly the two descriptions agree as to the direction of the gravitational force, since $-\mathbf{e}_r$, being directed towards the centre of the Earth, is always 'vertically downwards' from a local point of view. It remains then to compare the two expressions for the magnitude.

Exercise 15

(i) Show that $g = MG/R^2$, where R is the radius of the Earth.

(ii) Given that $g = 9.81\,\mathrm{m\,s^{-2}}$, $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ and $R = 6.4 \times 10^6\,\mathrm{m}$, estimate the mass M of the Earth.

[*Solution on page 50*]

It can be shown that the model of a constant gravitational force is quite a reasonable one for heights of a few kilometres above the Earth's surface. Consider two points P_1 and P_2 lying on the same radial line, with P_1 on the surface and P_2 at a distance h above it (see Figure 9). We wish to compare the force of gravity at points P_1 and P_2 on a particle of mass m.

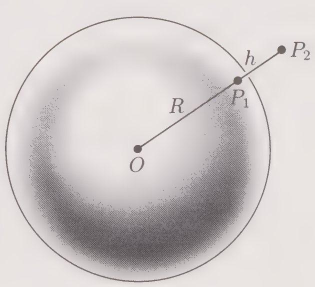


Figure 9

At both points the force on m is 'downward', along the direction of $-\mathbf{e}_r$. The magnitude of the force at P_1 on the Earth's surface (at radius R) is

$$F_1 = \frac{mMG}{R^2}.$$

At the point P_2, which is a height h above P_1, the force has magnitude

$$F_2 = \frac{mMG}{(R+h)^2}.$$

The ratio of these two magnitudes is

$$\frac{F_2}{F_1} = \frac{R^2}{(R+h)^2} = \frac{1}{(1+h/R)^2}.$$

Provided that h is small compared to R, the force of gravity at point P_2 is nearly equal to that at P_1.

Exercise 16

Given that the acceleration due to gravity is $9.81\,\mathrm{m\,s^{-2}}$ at the Earth's surface, and that the radius of the Earth is 6.4×10^6 m, estimate the acceleration due to gravity at a height of 10^4 m above the Earth's surface.

[*Solution on page 50*]

Summary of Section 2

1. **Kepler's laws** of planetary motion are as follows.

   (I) Each planet moves in an ellipse, with the Sun at one focus.

   (II) The line joining a planet to the Sun sweeps out equal areas in equal times.

   (III) The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

2. (i) The **ellipse** shown in Figure 10 has *semi-major axis* a and *semi-minor axis* b. With respect to the axes $O'XY$, its equation is

   $$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1,$$

   where $a \geq b > 0$.

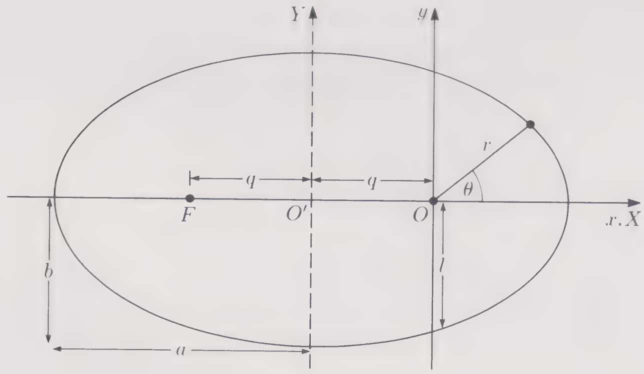


Figure 10

   (ii) The area of the ellipse is πab.

   (iii) The *foci* of the ellipse are located a distance q to either side of the centre O', where

   $$q = \sqrt{a^2 - b^2}.$$

   (iv) With respect to axes Oxy with origin at the right-hand focus, and in polar coordinates, the equation of the ellipse is

   $$\frac{l}{r} = 1 + e\cos\theta,$$

where

$$l = \frac{b^2}{a}$$

is the *semi-latus rectum*, and

$$e = \frac{q}{a} = \sqrt{1 - \frac{b^2}{a^2}}$$

is the *eccentricity*, where $0 \leq e < 1$.

3. **Newton's universal law of gravitation** states that the gravitational force on a particle with mass m_i and position vector $\mathbf{r}_i$ due to another particle with mass m_j and position vector $\mathbf{r}_j$ is

$$\mathbf{F}_{ij} = \frac{Gm_i m_j}{|\mathbf{r}_j - \mathbf{r}_i|^3}(\mathbf{r}_j - \mathbf{r}_i),$$

where G is the *gravitational constant*.

4. A body has a **spherically symmetric** mass distribution if (with the origin of coordinates at the centre of the body) its mass density function $\rho(\mathbf{r})$ depends only on the radial distance $|\mathbf{r}| = r = \sqrt{x^2 + y^2 + z^2}$. The gravitational force exerted by such a body B on an exterior particle with mass m and position vector $\mathbf{r}$ is given by

$$\mathbf{F}(\mathbf{r}) = -\operatorname{grad} U(\mathbf{r}) = -\frac{GmM}{r^2}\mathbf{e}_r,$$

where $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector in the outward radial direction,

$$U(\mathbf{r}) = -\frac{GmM}{r}$$

is the gravitational potential energy of the particle, and

$$M = \int_B \rho(\mathbf{r}')\, dV$$

is the total mass of the body.

3 Orbits

In Subsection 2.2 Newton's universal law of gravitation was introduced, leading to a derivation of the corresponding gravitational potential energy function for a spherically symmetric mass distribution. In the current section we shall use this result in developing a mathematical model to describe the orbit of a body under the gravitational attraction of a much larger body. To a reasonably good approximation, this model applies to the orbits of a planet around the Sun, of a moon around a planet, of a man-made satellite around a planet or around the Sun, and of comets around the Sun. The model will enable us to derive Kepler's three laws of planetary motion, which were stated and explained in Subsection 2.1.

Kepler's laws are also the first item in the Summary of Section 2 on the previous page.

The modelling assumptions which suffice to yield these laws are as follows.

Modelling assumptions for gravitational orbits

(i) The world-view of Newton is valid, including his three laws of mechanics and his universal law of gravitation.

(ii) All forces other than the gravitational attraction between the two bodies may be neglected. (In particular, the gravitational effect of other celestial objects is ignored.)

(iii) One body is much more massive that the other.

(iv) The more massive body has a spherically symmetric distribution of mass.

Of these assumptions, (i) is the hardest to improve upon and (iii) is the easiest, while (ii) and (iv) are intermediate. It was only with Einstein's general theory of relativity in the early twentieth century that small corrections to Newton's world-view (Assumption (i)) were introduced. This was done in order to explain small departures from the Newtonian prediction that had been observed in the orbit of the planet Mercury. We could (but will not!) improve upon Assumption (iii) using the mathematics developed in the course. This improvement is required, for example, to describe the orbits of two stars of comparable mass under their mutual gravitational attraction.

3.1 Central forces

By Assumption (iii) above, we suppose that the more massive body (of mass M) does not move at all under the gravitational action of the lighter body (of mass m). Its centre of mass coincides with its geometric centre, by Assumption (iv). For convenience, we choose the origin of coordinates to be at this centre, as shown in Figure 1.

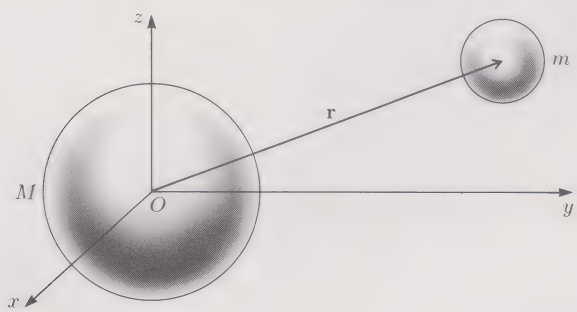


Figure 1

It was shown in *Unit 17* that the centre of mass of a system of particles acted upon by external forces moves as if all the mass of the system were concentrated there. We may therefore treat the motion of the centre of mass of the lighter body as if it were a particle. Furthermore, since the heavier body is assumed to be spherically symmetric (in gravitational terms), we can apply the result of Subsection 2.2 which equates the gravitational effect of such a body with that of a particle of mass M at its centre. The resulting two-particle view is shown in Figure 2, where $\mathbf{r}$ is the position vector of the lighter particle.

Unit 17 Subsection 2.3

A similar argument to this was employed in the solution to Exercise 14 in Section 2.

Equation (19) of Section 2.

There is a simplifying feature of this model which can be shown at the outset: *the motion of m occurs in a plane.* The proof of this statement rests on the fact that the gravitational force exerted on m is a *central force.* In general, this term describes a force which is directed from the particle m towards or away from the source of the force. If the source is at the origin, this means that a central force has the direction of $-\mathbf{r}$ or of $\mathbf{r}$. The definition is as follows.

As earlier in the unit, we use the symbol m as a convenient label for the particle of mass m.

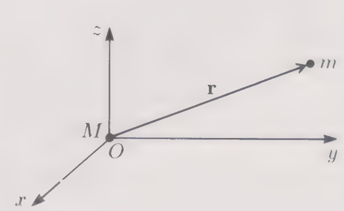


Figure 2

> A **central force F**, acting from the origin on a particle with position vector $\mathbf{r}$, has the form
>
> $$\mathbf{F}(\mathbf{r}) = g(\mathbf{r})\mathbf{r}, \qquad (1)$$
>
> where $g(\mathbf{r}) = g(x, y, z)$ is a scalar function of position.

Exercise 1

Which of the following are central forces (where K is a constant)?

(a) $\mathbf{F}(\mathbf{r}) = K\mathbf{r}$

(b) $\mathbf{F}(\mathbf{r}) = \dfrac{K}{r^2}\mathbf{r}$

(c) $\mathbf{F}(\mathbf{r}) = \dfrac{Kx}{r^2}\mathbf{r}$

(d) $\mathbf{F}(\mathbf{r}) = \dfrac{Ky}{r^2}\mathbf{j}$

(e) $\mathbf{F}(\mathbf{r}) = \dfrac{K}{r^2}(\mathbf{i}+\mathbf{j}+\mathbf{k})$

(f) $\mathbf{F}(\mathbf{r}) = \dfrac{K}{r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

Exercise 2

What is the scalar function $g(\mathbf{r})$ for the gravitational system of Figure 2?

[*Solutions on page 50*]

The angular momentum $\mathbf{L}$ of a particle with mass m and position vector $\mathbf{r}$ relative to the origin is, as defined in *Unit 29*,

Unit 29 Subsection 1.2

$$\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}. \tag{2}$$

In *Unit 29* we used the symbol $\boldsymbol{l}$ for the angular momentum of a single particle and $\mathbf{L}$ for the angular momentum of a system of particles. Throughout this unit we refer only to the angular momentum of a single particle, and for visual convenience this is represented by the symbol $\mathbf{L}$.

This definition is not confined to two-dimensional motion, though the main application in *Unit 28* was to motion in a circle. However, it is a consequence of the results of Exercise 3 below that if the particle is acted upon by a *central* force then its motion *does* lie in a plane.

Exercise 3

Show that:

(i) when the angular momentum $\mathbf{L}$ is non-zero, the position vector $\mathbf{r}$ is always perpendicular to $\mathbf{L}$;

(ii) when the force on the particle of mass m is a central force then $\dot{\mathbf{L}} = \mathbf{0}$, so that $\mathbf{L}$ is a constant vector;

(iii) when $\mathbf{L} = \mathbf{0}$, the velocity $\dot{\mathbf{r}}$ and position vector $\mathbf{r}$ are parallel or anti-parallel if neither is zero.

[*Solution on page 50*]

From parts (i) and (ii) of Exercise 3, we can say that if $\mathbf{L} \neq \mathbf{0}$ then the position vector $\mathbf{r}$ of the particle of mass m is always perpendicular to the fixed direction defined by the constant angular momentum vector $\mathbf{L}$. In the general case then, when $\mathbf{L}$ is non-zero, the particle acted upon by a central force moves in the plane perpendicular to the fixed direction of $\mathbf{L}$, as shown in Figure 3. Also (from part (iii)), in the event that $\mathbf{L}$ is zero, the velocity lies along a straight line from the origin, and the particle will continue to move along this straight line.

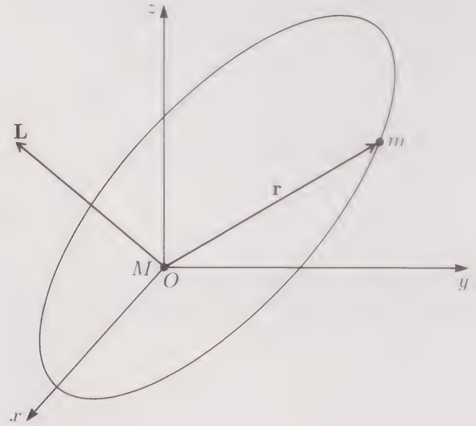


Figure 3

The constancy of the angular momentum $\mathbf{L}$ for a particle in gravitational orbit arises from the fact that gravity is a central force, and not from any other detail of its form. This result applies also, therefore, to other central forces.

Motion under a central force

A particle with mass m and position vector $\mathbf{r}$, which moves solely under the action of a central force, has constant angular momentum

$$\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}. \tag{2}$$

Its motion is confined to the plane perpendicular to the direction of $\mathbf{L}$ when $\mathbf{L}$ is non-zero and to a straight line if $\mathbf{L} = \mathbf{0}$.

As the motion is planar, we can now choose to orient the coordinate axes so that the motion lies in the (x, y)-plane, as indicated in Figure 4. This will permit us to apply the apparatus of plane polar coordinates which was developed in Subsection 1.3.

It has yet to be established that the planar motion of a particle in gravitational orbit is as simple as that depicted here. For example, some other central forces lead to spiralling motions.

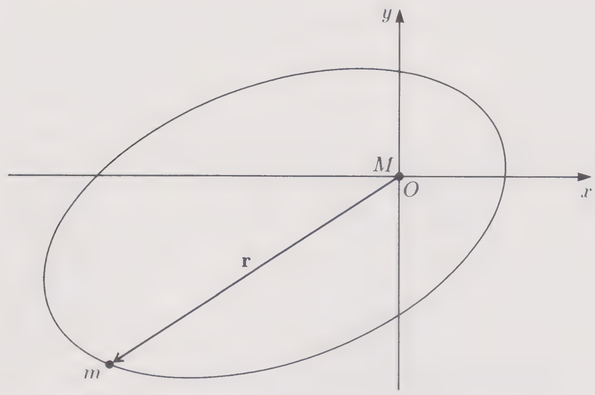


Figure 4

Since the angular momentum $\mathbf{L}$ is a constant vector, both of its direction and magnitude are constant. The constancy of its *direction* sufficed above to show that the orbit always lies in a plane. This is actually *part* of the content of Kepler's first law, which asserts that the orbits are ellipses and therefore plane figures.

We have not yet used the additional information that the *magnitude* of $\mathbf{L}$ is constant, and this is actually sufficient to prove Kepler's second law. As you may recall from Subsection 2.1, this law relates to the rate at which area A is swept out by the line joining a planet to the Sun (which corresponds here to the line of the position vector $\mathbf{r}$). We showed that $\dot{A}$ was related to the radial distance r and the angular velocity $\dot{\theta}$ by the equation

$$\dot{A} = \tfrac{1}{2}r^2\dot{\theta}.$$

Equation (9) of Section 2

Kepler's second law states that $\dot{A}$ is constant, which from above is equivalent to the condition

$$\frac{d}{dt}(r^2\dot{\theta}) = 0. \tag{3}$$

In the exercise below you are asked to show that Kepler's second law is a consequence of the constancy of angular momentum.

Exercise 4

(i) By expressing $\mathbf{r}$ and $\dot{\mathbf{r}}$ in terms of the plane polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ (Equations (8) and (12) of Section 1), show that the angular momentum of a particle moving in the (x, y)-plane is $\mathbf{L} = L_z\mathbf{k}$, where

$$L_z = mr^2\dot{\theta}.$$

(ii) Deduce that if $\dot{L}_z = 0$ then Equation (3) is satisfied.

[*Solution on page 50*]

In summary, thus far, we can say the following.

> Kepler's second law, and the fact that each planet moves in a plane, are consequences only of the fact that the gravitational force between the planet and the Sun is a central force; taken together, they amount to a statement of the constancy of angular momentum.

3.2 Isotropic central forces and gravity

With the force of gravity in mind as the ultimate target, we now specialize from central forces to the more restricted case of *isotropic* central forces. These are central forces for which the magnitude depends only on the radial distance from the origin.

> An **isotropic** central force has the form
>
> $$\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r, \qquad (4)$$
>
> where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin, and $\mathbf{e}_r = \mathbf{r}/r$.

A central force was defined by $\mathbf{F}(\mathbf{r}) = g(\mathbf{r})\mathbf{r}$, which is equivalent to

$$\mathbf{F}(\mathbf{r}) = g(\mathbf{r})r\mathbf{e}_r.$$

This force is also isotropic if $g(\mathbf{r}) = g(r)$.

Exercise 5

(i) Which of the forces listed in Exercise 1 (page 28) are isotropic central forces?

(ii) Show that the gravitational force is isotropic, and find the function $f(r)$ for this case.

[*Solution on page 50*]

Choosing our axes as in Figure 4, so that the orbit lies in the (x, y)-plane, we have $z = 0$ and $r = \sqrt{x^2 + y^2}$. Newton's second law, for the motion of a particle of mass m acted upon by an isotropic central force, is

$$m\ddot{\mathbf{r}} = f(r)\mathbf{e}_r.$$

Since the right-hand side of this equation is directed along the unit vector $\mathbf{e}_r$, it makes sense to express the acceleration $\ddot{\mathbf{r}}$ in terms of plane polar coordinates, using Equation (13) of Section 1. The particle's equation of motion in the plane can then be written as

$$m(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta})\mathbf{e}_\theta = f(r)\mathbf{e}_r,$$

from which we have

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \qquad (5)$$

and

$$\frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0.$$

The second equation is equivalent to Equation (3), whose validity was established for any central force in Exercise 4. The consequent constancy of the angular momentum magnitude,

$$L_z = mr^2\dot{\theta}, \qquad (6)$$

can be exploited to eliminate the variable $\dot{\theta}$ in Equation (5). Thus on putting $\dot{\theta} = L_z/mr^2$ into Equation (5), we obtain

$$m\ddot{r} - mr\left(\frac{L_z}{mr^2}\right)^2 = f(r)$$

or

$$m\ddot{r} = f(r) + \frac{L_z^2}{mr^3}. \qquad (7)$$

Equations (6) and (7) contain, between them, the possibility of a complete solution to the problem of finding the orbit of a particle of mass m under the action of any isotropic central force, given by Equation (4). The procedure to do this would be as follows.

(i) The initial conditions are the values of $[r, \theta]$ and $[\dot{r}, \dot{\theta}]$ at some initial time, say $t = 0$.

(ii) Since L_z is a constant, these initial conditions determine the value of L_z for all subsequent times, from Equation (6).

(iii) Equation (7) may be solved, either analytically if possible or numerically, to obtain an expression or values for $r(t)$, where $t > 0$.

(iv) This solution may be substituted into Equation (6) to obtain an expresion for $\dot{\theta}(t)$, from which $\theta(t)$ can be found by direct integration.

The approach we shall eventually adopt, however, is somewhat different. Instead of seeking $r(t)$ and $\theta(t)$ individually, which are parametric equations for the orbit, we look for a direct relationship $r = r(\theta)$. This may describe an orbit which is *bound* or *unbound*. For a bound orbit (see curve (a) in Figure 5), the distance of the particle from the origin O is bounded for all time. For an unbound orbit (see curve (b) in Figure 5), the particle approaches from infinitely far away, is deflected by the force centred at O and departs again to infinite distances. Corresponding to the bound orbit shown in Figure 6(a), the function $r(\theta)$ has the graph of Figure 6(b).

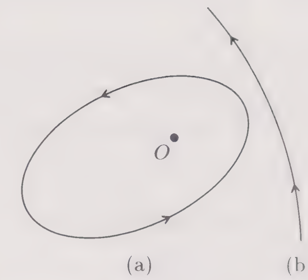


Figure 5

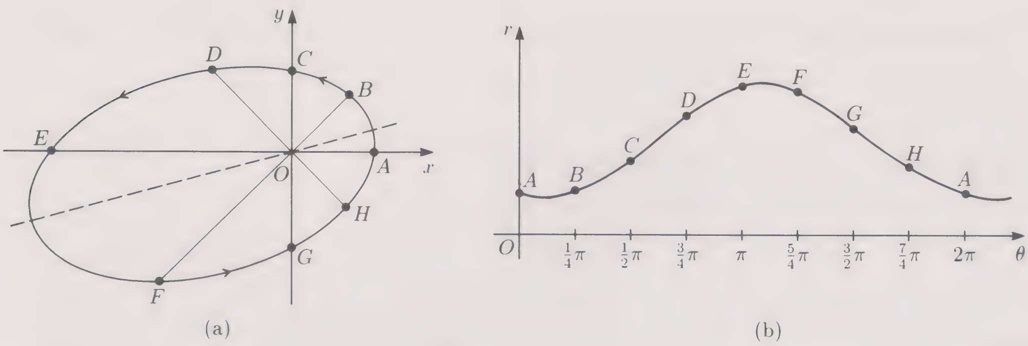


Figure 6

Thus far, the results obtained relate to motion under any isotropic central force, and not just to gravity. At this point we shall finally specialize to the case of gravitation, though the results may also be applied to any other force which is described by a similar inverse square law.

The electrostatic force is one example of an isotropic central force. You considered a two-dimensional example of such a force in Exercise 15 of Section 1.

Specializing to gravity means using the fact that for this particular isotropic central force we have (from Exercise 5(ii))

$$f(r) = -\frac{GmM}{r^2},$$

so that Equation (7) for the particle motion becomes

$$m\ddot{r} = \frac{L_z^2}{mr^3} - \frac{GmM}{r^2}. \tag{8}$$

This equation looks somewhat formidable, but it can be tamed. In Subsection 3.3 you will see how a relationship $r = r(\theta)$ can be obtained from Equation (8) to describe gravitational orbits.

For the remainder of this subsection we shall investigate what can be deduced about the motion without solving Equation (8) directly. This indirect approach depends on the constancy of both the angular momentum L_z and the total mechanical energy, E.

It follows from the result of Exercise 13 of Section 1 that, for a particle moving in two dimensions and acted upon by the (conservative) gravitational force, the constant total mechanical energy E may be written as

$$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r), \tag{9}$$

where $U(r) = -GmM/r$ is the gravitational potential energy function. From Equation (6), the constant angular momentum is given by $L_z = mr^2\dot{\theta}$. After using this equation to replace $\dot{\theta}$ by $L_z/(mr^2)$, Equation (9) becomes

Equation (19) of Section 2

We write $U(r)$ rather than $U(\mathbf{r})$ here to emphasize that the gravitational potential energy U depends only upon the radial distance r.

$$E = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\left(\frac{L_z}{mr^2}\right)^2 - \frac{GmM}{r}$$

or $$E = \tfrac{1}{2}m\dot{r}^2 + \left(\frac{L_z^2}{2mr^2} - \frac{GmM}{r}\right). \tag{10}$$

Hence, by using the constancy of the angular momentum, we have expressed the total mechanical energy in terms of $\dot{r}$ and r only.

In fact, any isotropic central force is conservative, and has a constant total mechanical energy E given by Equation (9). You will see this shown in Section 4.

As a result, Equation (10) has the appearance of an energy equation for a particle moving in *one dimension*. The first term, $\frac{1}{2}m\dot{r}^2$, resembles the one-dimensional kinetic energy. Since the bracketed term in Equation (10) looks like a one-dimensional potential energy function, we shall call it the *effective potential energy* and denote it by $U^{\text{eff}}(r)$. This gives

As a check, if Equation (10) is differentiated with respect to time (with $\dot{E} = 0$), then the outcome is Equation (8). Put another way, Equation (10) is a first integral of Equation (8).

$$E = \tfrac{1}{2}m\dot{r}^2 + U^{\text{eff}}(r),$$

where the effective potential energy is

$$U^{\text{eff}}(r) = \frac{L_z^2}{2mr^2} - \frac{GmM}{r}. \tag{11}$$

In summary, then, the *radial* motion of the particle is as if it were moving in one dimension, with $r > 0$, under the influence of a one-dimensional conservative force having potential energy function $U^{\text{eff}}(r)$. From this insight you might expect that Equation (8) for the radial motion could be written as

$$m\ddot{r} = -\frac{d}{dr}U^{\text{eff}}(r),$$

by analogy with the definition of a one-dimensional potential energy function. This is indeed the case, as can be verified by differentiating Equation (11).

Equation (1) of Section 1

Obviously, the particle whose orbit we seek is not really moving in a single dimension, since as r varies with time the angle θ is changing too. However, we have managed in Equation (10) to express the total mechanical energy of the orbiting particle in *the same format* as that for a one-dimensional motion. This means that we can deduce information about the radial coordinate of the orbit, using the effective potential energy function, with the approach that was adopted in Subsection 1.1 for an actual one-dimensional motion.

You saw in the analysis of Subsection 1.1 that whether a motion is bound or not depends both on the shape of the graph of the potential energy function and upon the value of the constant E. Here the effective potential energy is given by Equation (11), and has the graph shown in Figure 7. The main features of the function U^{eff} illustrated here are as follows.

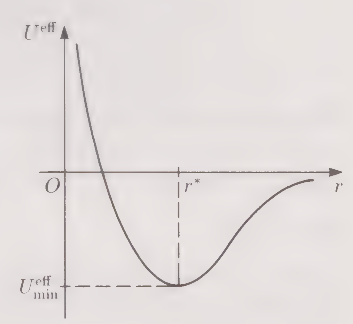


Figure 7

(i) When r is small, the term $L_z^2/(2mr^2)$ dominates the term $-GmM/r$, so $U^{\text{eff}}(r)$ becomes large and positive as r approaches zero.

(ii) When r is large, both of the terms in $U^{\text{eff}}(r)$ tend towards zero, so that the limit of $U^{\text{eff}}(r)$ for large values of r is zero.

(iii) For a range of intermediate values of r, the term $-GmM/r$ dominates the term $L_z^2/(2mr^2)$, causing $U^{\text{eff}}(r)$ to be negative with a minimum value at some point $r = r^*$.

Exercise 6

Find the value $r = r^*$ for which U^{eff} has a minimum, and calculate the corresponding minimum value $U^{\text{eff}}_{\text{min}} = U^{\text{eff}}(r^*)$.

[*Solution on page 50*]

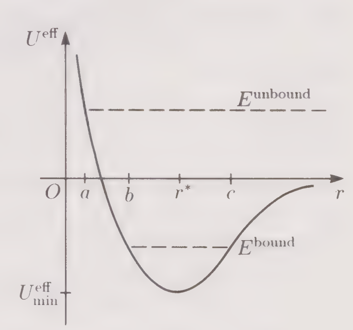


Figure 8

Figure 8 shows additions to the graph of U^{eff} relating to two values of the total mechanical energy E, which correspond to a bound and an unbound motion. A particle approaching the origin O with some *positive* energy, E^{unbound}, reaches a closest point to O at $r = a$, following which it is always moving away towards larger values of r. On the other hand, a particle with some *negative* energy, E^{bound}, undergoes a periodic oscillation of the radial coordinate between a minimum value $r = b$ and a maximum value $r = c$. (In Subsection 1.1 these would have been bound and unbound motions along a line. Here they correspond to bound and unbound orbits.)

Exercise 7

Describe the orbit corresponding to the total mechanical energy $E = U^{\text{eff}}_{\text{min}}$.

[*Solution on page 50*]

The following summary details what information has been obtained so far about orbital motion under the influence of gravity.

Gravitational orbits

Consider the planar gravitational orbit of a particle, with mass m and polar coordinates $[r, \theta]$, due to a spherically symmetric body of much larger mass M, situated at the origin.

(i) The angular momentum L_z of the particle is constant, where

$$L_z = mr^2\dot{\theta}. \tag{6}$$

(This is a property of any central force.)

(ii) The total mechanical energy E of the particle is constant. It is given by

(a) $$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r), \tag{9}$$

where $U(r) = -GmM/r$ is the (actual) potential energy of the particle;

(b) $$E = \tfrac{1}{2}m\dot{r}^2 + U^{\text{eff}}(r),$$

where

$$U^{\text{eff}}(r) = \frac{L_z^2}{2mr^2} - \frac{GmM}{r} \tag{11}$$

is the effective potential energy.

(iii) The orbit is unbound if $E \geq 0$, and bound if $U^{\text{eff}}_{\text{min}} \leq E < 0$, where $U^{\text{eff}}_{\text{min}}$ is the minimum value of $U^{\text{eff}}(r)$.

(iv) The equation of motion for $r(t)$ is

$$m\ddot{r} = \frac{L_z^2}{mr^3} - \frac{GmM}{r^2}. \tag{8}$$

The properties of gravitational orbits arising from the constancy of angular momentum and of total mechanical energy can be applied to obtain useful information in specific cases, as illustrated by the exercises below.

Exercise 8

A particle of mass m is launched with speed v at right angles to the line which joins it to a spherically symmetric body of much larger mass M. At the instant of launch, the distance between m and the centre of M is R.

(i) Find the angular momentum of the particle for all subsequent times.

(ii) Find the total mechanical energy of the particle for all subsequent times.

(iii) Show that the motion will be bound provided that $v^2 < 2MG/R$.

Exercise 9

A particle of mass m is launched as described in Exercise 8. Its speed of launch v and distance of launch R are related by the condition $v^2 = MG/(2R)$.

(i) Find the total mechanical energy, and show that the resulting orbit is bound.

(ii) Find r_{min} and r_{max}, the respective smallest and largest distances of the particle from the centre of the attracting mass M. (Hint: From the solution to Exercise 8, the total mechanical energy is $E = \frac{1}{2}mv^2 - GmM/R$, but this is also expressible as $E = \frac{1}{2}m\dot{r}^2 + U^{\text{eff}}(r)$ for all values of r and $\dot{r}$ on the particle's trajectory. At the turning points, when $r = r_{\text{min}}$ or $r = r_{\text{max}}$, we have $\dot{r} = 0$. Hence the equation for the turning points is $E = U^{\text{eff}}(r)$.)

Here r_{min} and r_{max} correspond to the values b and c of Figure 8.

[*Solutions on page 50*]

3.3 The equation of gravitational orbits

We now revert to consideration of the radial equation of gravitational motion,

$$m\ddot{r} = \frac{L_z^2}{mr^3} - \frac{GmM}{r^2}. \tag{8}$$

As mentioned earlier, we seek to obtain from this equation a relation of the form $r = r(\theta)$ to describe gravitational orbits. To do so, we apply the chain rule of differentiation to convert Equation (8), which specifies a function $r(t)$, to an equation for $r(\theta)$. For the first derivative, $\dot{r}$, we have

$$\dot{r} = \frac{dr}{dt} = \frac{d\theta}{dt}\frac{dr}{d\theta} = \dot{\theta}\frac{dr}{d\theta}.$$

From Equation (6), $\dot{\theta}$ can be replaced by $L_z/(mr^2)$, where L_z is a constant. The previous equation can then be expressed as

$$\dot{r} = \frac{L_z}{mr^2}\frac{dr}{d\theta}$$

or, in a form which will shortly be convenient, as

$$\dot{r} = -\frac{L_z}{m}\frac{d}{d\theta}\left(\frac{1}{r}\right). \tag{12}$$

The equation of motion contains $\ddot{r}$, which may be written in terms of a derivative with respect to θ by a further application of the chain rule. Thus, we obtain

$$\ddot{r} = \frac{d}{dt}\dot{r} = \frac{d\theta}{dt}\frac{d}{d\theta}\dot{r} = \dot{\theta}\frac{d}{d\theta}\dot{r}$$

which, after another use of the relation $\dot{\theta} = L_z/(mr^2)$ and substitution for $\dot{r}$ from Equation (12), becomes

$$\ddot{r} = -\frac{L_z^2}{m^2r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right).$$

Equation (8) can now be written as

$$-\frac{L_z^2}{mr^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{L_z^2}{mr^3} - \frac{GmM}{r^2}.$$

Defining

$$u(\theta) = \frac{1}{r(\theta)} \tag{13}$$

gives, after some rearrangement,

$$\frac{d^2u}{d\theta^2} + u(\theta) = \frac{m^2MG}{L_z^2}. \tag{14}$$

Equations (13) and (14) contain the information necessary to deduce the orbit $r = r(\theta)$. You have met equations like Equation (14) many times. It is a second-order, constant-coefficient, inhomogeneous differential equation, whose general solution has two arbitrary constants. One way of writing the solution is

For example, in *Unit 7* Subsection 2.4 a similar equation was derived when modelling the motion of a particle suspended from a perfect spring in a uniform gravitational field.

$$u(\theta) = B\cos(\theta - \theta_0) + \frac{m^2MG}{L_z^2}. \tag{15}$$

The two arbitrary constants in this solution are the amplitude B and the phase θ_0. (We assume that B is non-negative. This represents no loss of generality, since the sign of the term $B\cos(\theta - \theta_0)$ may be changed by adding π to the arbitrary phase θ_0.) The orbital equation is obtained from Equation (15) on using Equation (13) to replace u by $1/r$. After multiplying through by

$$l = \frac{L_z^2}{m^2MG} \tag{16}$$

and putting $e = Bl$, the orbital equation can be written as

Here B is a non-negative but otherwise arbitrary constant, so the same is true of e.

$$\frac{l}{r(\theta)} = 1 + e\cos(\theta - \theta_0). \tag{17}$$

Equations (16) and (17), together with the knowledge that L_z (or, equivalently, l) and e are constants, provide the required mathematical description of gravitational orbits for the model outlined at the beginning of this section. The remainder of the section will be spent in examining the implications of these results.

Note first that the introduction of the symbols l and e has permitted us to write the orbital equation in a form which is identical to Equation (8) of Section 2. Provided that $e < 1$, therefore, Equation (17) represents an *ellipse* with eccentricity e and semi-latus rectum l.

The terms 'eccentricity' and 'semi-latus rectum' are used to describe e and l whether or not the orbit is an ellipse.

Exercise 10

Equation (17) describes a general gravitational orbit.

(i) By considering the possible values of the right-hand side of this equation, show that the distance of closest approach of the particle of mass m to the origin occurs at the angle $\theta = \theta_0$, and has the value

$$r_{\text{min}} = \frac{l}{1+e}.$$

(ii) Show that, when $e < 1$, the maximum distance of the particle of mass m from the origin occurs at the angle $\theta = \theta_0 + \pi$, and has the value

$$r_{\text{max}} = \frac{l}{1-e}.$$

(iii) Show that when $e \geq 1$ there is no maximum value of r.

(iv) From parts (i)–(iii), give conditions on the eccentricity e which characterize bound and unbound orbits.

[*Solution on page 51*]

From the results of Exercise 10 we can sketch the possible types of orbit that Equation (17) represents. This is done in Figure 9 below. When $e < 1$, the orbits are closed (ellipses) with closest and furthest distances r_{min} and r_{max} respectively. When $e \geq 1$, the orbits are unbound (hyperbolas or parabolas), and only the distance of closest approach r_{min} is defined. In either case, the orbit is symmetrical about the line $\theta = \theta_0$.

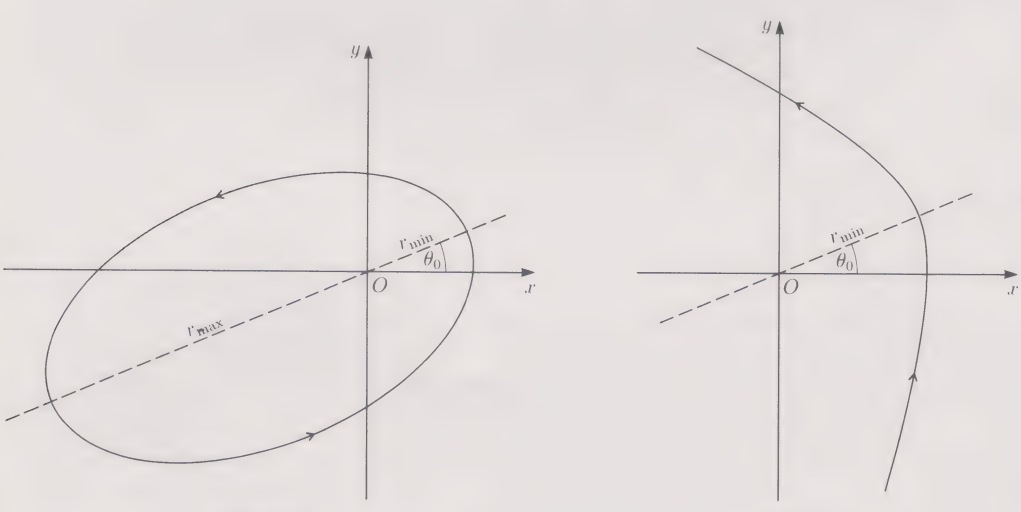
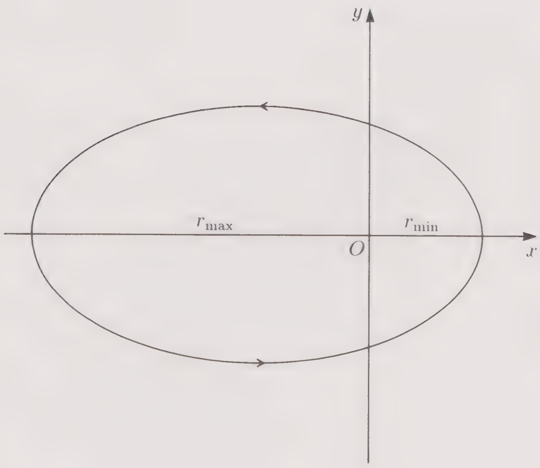


Figure 9

It is convenient to consider the gravitational orbits with respect to a set of axes which are rotated anticlockwise by an angle θ_0 from those of Figure 9. This is equivalent to choosing $\theta_0 = 0$ in the orbital equation, or to choosing the x-axis to coincide with an orbital symmetry, and represents no loss of generality. In place of Equation (17) we then have

$$\frac{l}{r} = 1 + e\cos\theta, \tag{18}$$

where the semi-latus rectum l is given as before by Equation (16). The corresponding pictures for the orbits are shown in Figure 10.



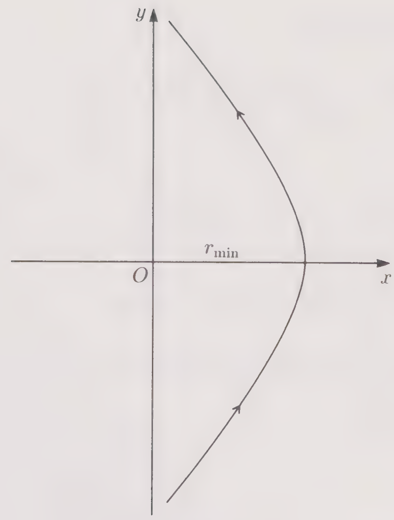


Figure 10

Equation of gravitational orbits

The Newtonian model for the gravitational orbit of a body of mass m around a spherically symmetric body of much larger mass M located at the origin leads to the orbital equation

$$\frac{l}{r} = 1 + e\cos\theta, \tag{18}$$

where the eccentricity e is non-negative and the semi-latus rectum l is given by

$$l = \frac{L_z^2}{m^2MG}. \tag{16}$$

Here L_z is the constant angular momentum magnitude of the orbiting body. If $0 \leq e < 1$ then Equation (18) describes an ellipse.

Values for the constants l and e in Equation (18) can be calculated in a specific case from experimental observations, and this can be done in various ways. For example, suppose that the distance of closest approach r_{min} and the corresponding angular velocity $\dot{\theta}$ have been measured, and that the value of the central mass M is known. Then a value for L_z/m can be obtained from Equation (6). This leads in turn to a value for l from Equation (16) and a value for e from the result of Exercise 10(i).

The energy analysis carried out in Subsection 3.2 predicted that gravitational motion will be bound if the total mechanical energy E satisfies the inequality $U_{\text{min}}^{\text{eff}} \leq E < 0$, and unbound if $E \geq 0$. It is to be expected that these predictions are related to those obtained in terms of the eccentricity e in Exercise 10, namely, that an orbit is bound if $e < 1$ and unbound if $e \geq 1$. We consider this relationship in the example below.

This analysis was based on Equations (10) and (11), together with Figure 8.

Example 1

Show that the total mechanical energy can be written as

$$E = \frac{mMG}{2l}(e^2 - 1).$$

Hence verify that the condition $E \geq 0$ for unbound orbits is equivalent to $e \geq 1$, and that the condition $E < 0$ for bound orbits is equivalent to $e < 1$ (given that $e \geq 0$).

Solution

From Equation (10), E can be written as

$$E = \tfrac{1}{2}m\dot{r}^2 + \frac{L_z^2}{2mr^2} - \frac{GmM}{r}.$$

Using Equation (12) for $\dot{r}$, this becomes

$$E = \tfrac{1}{2}m\frac{L_z^2}{m^2}\left[\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]^2 + \frac{L_z^2}{2mr^2} - \frac{GmM}{r}.$$

From Equation (16) for l, this is

$$E = \frac{mMG}{2l}\left\{\left[\frac{d}{d\theta}\left(\frac{l}{r}\right)\right]^2 + \left(\frac{l}{r}\right)^2 - 2\left(\frac{l}{r}\right)\right\}.$$

But l/r is given in terms of θ by Equation (18), so that

$$E = \frac{mMG}{2l}\left\{e^2\sin^2\theta + (1+e\cos\theta)^2 - 2(1+e\cos\theta)\right\}.$$

On simplifying, this gives

$$E = \frac{mMG}{2l}(e^2-1).$$

So the condition $E \geq 0$ for unbound orbits corresponds to $e \geq 1$, and the condition $E < 0$ for bound orbits corresponds to $e < 1$. (Although this outcome is expected, it represents a useful cross-check on the results achieved.) □

Exercise 11

Describe the orbit given by Equation (18) when the eccentricity e is zero. What are r_{max} and r_{min} in this case, and what is the total mechanical energy?

Exercise 12

A satellite of mass m follows a circular orbit with constant angular velocity and radius d around a spherically symmetric planet having a much larger mass M.

(i) Show, from first principles (using Equation (13) of Section 1), that the satellite's acceleration lies along the radius of its orbit and has magnitude

$$|\ddot{\mathbf{r}}| = d\dot{\theta}^2.$$

(ii) Show too that its orbital angular momentum has magnitude

$$L_z = md^2\dot{\theta}.$$

(iii) By equating $m\ddot{\mathbf{r}}$ to the gravitational attraction on the satellite (Newton's second law) show that the value of the orbital radius agrees with that found in Exercise 11.

(iv) By adding the kinetic energy of the satellite to its gravitational potential energy, verify that the value of the total mechanical energy agrees with that found in Exercise 11.

[*Solutions on page 51*]

3.4 Planets and comets

In this brief subsection we shall verify the statement made at the beginning of this section to the effect that Kepler's laws of planetary motion are consequences of our model based on Newton's laws. Following this there is a short discussion about the orbits of comets.

Kepler's first law says that each planetary orbit is an ellipse, with the Sun at one focus. You showed in Exercise 3 that a planet's orbit lies in a plane because gravity is a *central* force. According to the model, a gravitational orbit within this plane has an equation of the form

$$\frac{l}{r} = 1 + e\cos\theta, \qquad (18)$$

where the semi-latus rectum l and the eccentricity e are positive constants for a particular planet.

Planetary orbits are clearly bound, and you established in Exercise 10 that $e < 1$ is a necessary and sufficient condition for bound motion. But for $e < 1$, Equation (18) is the equation developed in Subsection 2.1 to describe an ellipse with one focus at the origin. In our Newtonian model of planetary motion, the Sun is located at this focus. Thus Kepler's first law is a consequence of the model.

Equation (7) of Section 2

Kepler's second law says that the line joining a planet to the Sun sweeps out equal areas in equal times. We showed in Subsection 3.1 that this is equivalent to the equation

$$\frac{d}{dt}(r^2\dot{\theta}) = 0, \tag{3}$$

which expresses the constancy of the planet's angular momentum. This is again a consequence of gravity being a central force.

Kepler's third law is a relation between the period of a (bound) orbit and its lateral extent. It says that if T is the period and a is the semi-major axis of the elliptical orbit, then

$$T^2 = ka^3, \tag{19}$$

where k is a constant not depending on T or on a. In Exercise 13 below you are asked to show from basic principles that Equation (19) holds for the special case of a circular orbit, where the eccentricity e is zero. In this case the semi-major and semi-minor axes both equal the radius of the circle. In Exercise 14 you are then asked to establish Kepler's third law for any elliptical orbit.

Exercise 13

This exercise refers to Newton's quotation given at the beginning of Subsection 2.2.

(i) Show that the magnitude of the gravitational force between the Moon (of mass m) and the Earth (of mass M) is

$$\frac{GmM}{d^2},$$

where d is the separation between their centres.

(ii) Assume that the Moon is moving uniformly in a circle around the Earth, so that (from Exercise 12(i)) the magnitude of its acceleration is $|\ddot{\mathbf{r}}| = d\dot{\theta}^2$, where $\dot{\theta}$ is its constant angular velocity. Show that

$$|\ddot{\mathbf{r}}| = \frac{4\pi^2 d}{T^2},$$

where T is the period of the Moon's orbit.

(iii) Show that if air resistance is ignored then the acceleration of an apple dropped close to the surface of the Earth is

$$g = \frac{MG}{R^2},$$

where R is the Earth's radius.

(iv) From parts (i)–(iii) show that, according to the Newtonian model of gravitation,

$$T^2 = kd^3, \quad \text{where} \quad k = \frac{4\pi^2}{gR^2}.$$

(v) It was known by Newton that $d \simeq 60R$, $g \simeq 9.8\,\mathrm{m\,s^{-2}}$ and $T \simeq 28$ days, but it is now believed that he used for R, the Earth's radius, a value of 3500 'Italian miles', where one Italian mile is 5000 feet (rather than 5280 feet). This would make $R \simeq 5.3 \times 10^6$ m. Using these values (converted to appropriate SI units where necessary), do you find that the left- and right-hand sides of the predicted relation in part (iv) agree 'pretty nearly'? (A more accurate value for R is 6.4×10^6 m.)

Exercise 14

Consider an elliptical orbit with semi-major axis a and semi-minor axis b. Then it is known from Exercise 2 of Section 2 that the area of the ellipse is πab. Also, the semi-latus rectum is

$$l = \frac{b^2}{a} = \frac{L_z^2}{m^2MG}$$

(from Equation (5) of Section 2 and Equation (16) of this section), and the rate at which area is swept out is

$$\dot{A} = \tfrac{1}{2}r^2\dot{\theta} = \frac{L_z}{2m}$$

(from Equation (9) of Section 2 and Equation (6) of this section). Here L_z is the (constant) orbital angular momentum.

(i) From these facts, show that the orbital period, T, and the semi-major axis, a, are related by Kepler's third law, in the form

$$T^2 = ka^3, \quad \text{where} \quad k = \frac{4\pi^2}{MG}.$$

(ii) Show that, for a circular orbit around the Earth, this agrees with the result of Exercise 13(iv).

[*Solutions on page 51*]

Kepler's laws were formulated to describe the bound orbits of planets, so there is no reason to suppose that they should have significance for unbound orbits. The first and third laws can by their very nature apply only to the bound case. However, Kepler's second law applies even for unbound orbits since, as you have seen, it depends only upon the fact that gravity is a central force.

The Newtonian model which led to Equation (18) for gravitational orbits makes no assumption as to whether or not the motion is bound (with eccentricity $e < 1$) or unbound (with $e \geq 1$). As shown in Example 1, these conditions for the eccentricity are related to similar conditions on the total mechanical energy E of an orbit. If $E < 0$ for a body moving under the gravitational influence of the Sun, then the body has insufficient energy to 'escape' from the Sun's gravitational field and is confined to bound and periodic motion. Bodies with energy $E \geq 0$ may approach the Sun once, but are not 'captured' by it. These unbound orbits are called *parabolic* when $e = 1$ ($E = 0$) and *hyperbolic* when $e > 1$ ($E > 0$).

At any particular time there are hundreds of bodies in the solar system moving in nearly or actually unbound orbits under the Sun's gravity. These are the *comets.* Precise details of cometary orbits are not easily obtained, but it appears that most comets are in weakly bound orbits, with values of eccentricity e just less than unity. These comets therefore have highly *eccentric*, or flattened, orbits, as illustrated in Figure 11.

You investigated the connection between the eccentricity and shape of an ellipse in Exercise 4 of Section 2.

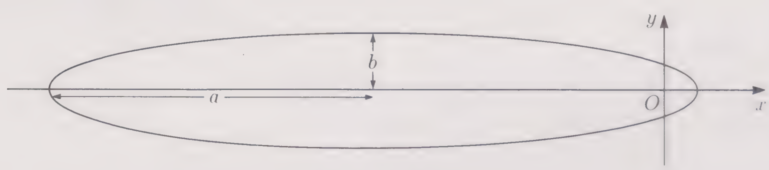


Figure 11

As a first approximation, the orbits of these bound comets obey Kepler's laws, but as they follow their elongated trajectories through the solar system they occasionally pass close to a planet. In these circumstances the gravitational effect of this *third* body can alter their orbits, even to the extent of raising their energies to positive values. Owing to its relatively large mass, Jupiter is a particular offender in this regard. The most famous bound comet is Halley's comet, which has a period of about 76 years and last passed close to the Sun in early 1986. Its orbit has eccentricity $e \simeq 0.97$ and a semi-major axis of about 18 times the distance from the Earth to the Sun. Comparable data for the planets, the Moon and Pluto are given in the table below.

For each body except the Earth, the values given in the table for mass, period, semi-major axis and mean diameter are multiples of the corresponding values for the Earth.

Name	Mass	Period of revolution around Sun	Semi-major axis of orbit	Eccentricity of orbit	Mean diameter
Sun	332 948.0				109.1
Moon	0.012			0.055	0.271
Mercury	0.055	0.241	0.387	0.206	0.391
Venus	0.815	0.615	0.723	0.007	0.951
Earth	5.974×10^{24} kg	365.256 days	149.6×10^{6} km	0.017	12 735 km
Mars	0.107	1.88	1.524	0.093	0.531
Jupiter	317.9	11.86	5.203	0.048	10.98
Saturn	95.1	29.46	9.539	0.056	9.09
Uranus	14.6	84.00	19.182	0.047	4.00
Neptune	17.2	164.79	30.058	0.009	3.78
Pluto	0.17	247.70	39.44	0.254	0.22

At any time there are comets in the solar system that are pursuing unbound trajectories. These bodies come in from distant places, pass partly around the Sun to a distance of closest approach and then recede, never to return to our solar system.

Summary of Section 3

1. A **central force F**, acting from the origin on a particle with position vector $\mathbf{r}$, has the form

$$\mathbf{F}(\mathbf{r}) = g(\mathbf{r})\mathbf{r},$$

where $g(\mathbf{r}) = g(x, y, z)$ is a scalar function of position. If the particle has mass m and is not acted upon by any other force, then it has constant angular momentum $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$. If the unit vector $\mathbf{k}$ is chosen in the direction of $\mathbf{L}$, then

(i) the motion of the particle is confined to the (x, y)-plane;

(ii) $\mathbf{L} = L_z\mathbf{k}$, where $L_z = mr^2\dot{\theta}$ is constant.

The gravitational force due to a spherically symmetric body whose centre is fixed at the origin is a central force. In this case, the constancy of L_z is equivalent to Kepler's second law.

2. An **isotropic** central force has the form

$$\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r,$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin, and $\mathbf{e}_r = \mathbf{r}/r$. For a particle of mass m acted upon by such a force, the radial equation of motion is

$$m\ddot{r} = f(r) + \frac{L_z^2}{mr^3},$$

which may be integrated (analytically or numerically) to give $r(t)$. The gravitational force is isotropic, with $f(r) = -GmM/r^2$, where M is the central mass and G is the gravitational constant. (It is assumed that M is much larger than m, in order that the central mass should remain fixed at the origin.)

3. For the particle moving under gravity as described above, the total mechanical energy E is constant. It may be expressed as

(i) $$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r),$$

where $U(r) = -GmM/r$ is the (actual) potential energy of the particle;

(ii) $$E = \tfrac{1}{2}m\dot{r}^2 + U^{\text{eff}}(r),$$

where

$$U^{\text{eff}}(r) = \frac{L_z^2}{2mr^2} - \frac{GmM}{r}$$

is the effective potential energy.

Orbits are unbound if $E \geq 0$, and bound if $U^{\text{eff}}_{\text{min}} \leq E < 0$, where $U^{\text{eff}}_{\text{min}}$ is the minimum value of $U^{\text{eff}}(r)$.

4. The equation of gravitational orbit for the particle described above is

$$\frac{l}{r} = 1 + e\cos\theta,$$

where the semi-latus rectum l is given by

$$l = \frac{L_z^2}{m^2MG}.$$

If $0 \leq e < 1$ then the orbit is an ellipse with eccentricity e, showing that Kepler's first law is satisfied. If $e \geq 1$ then the motion is unbound.

In terms of the eccentricity and the semi-latus rectum, the total mechanical energy is

$$E = \frac{GmM}{2l}(e^2 - 1).$$

5. Kepler's third law is also a consequence of this Newtonian model, since the period T of an elliptical orbit and its semi-major axis a are related by the equation

$$T^2 = ka^3, \quad \text{where} \quad k = \frac{4\pi^2}{MG}.$$

4 Central forces and conservation laws (Television Section)

4.1 Isotropic central forces

The television programme considers some general statements that can be made about the motion of a particle in three dimensions which is acted upon by a central force and, more especially, by an isotropic central force. When the force is *central*, then the angular momentum of the particle is constant and its motion lies in a plane. When in addition the force is *isotropic*, one can say further that there is a potential energy function for the particle, so that it has a constant total mechanical energy.

The first of these results was established in Subsection 3.1. The second was applied for the particular case of gravity in Subsection 3.2, but holds more generally. This more general version is a consequence of Result (i) below.

(i) If a scalar function $\phi(\mathbf{r})$ depends upon position only through the radial distance $|\mathbf{r}| = r = (x^2+y^2+z^2)^{1/2}$, so that $\phi(\mathbf{r}) = \phi(r)$, then the gradient of ϕ can be written as

$$\operatorname{grad}\phi = \frac{d\phi}{dr}\mathbf{e}_r,$$

where $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector in the outward radial direction.

(ii) Hence if $\mathbf{F}$ is an isotropic central force, that is

$$\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r$$

for some function f, then $\mathbf{F}$ is conservative, with a potential energy function $U(r)$ such that

$$f(r) = -\frac{dU}{dr}.$$

The motion of a particle in this force field is planar, and has constant total mechanical energy

$$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r).$$

Result (ii) follows from Result (i) on referring to the definition of a conservative force (Equation (3) of Section 1) and to the formula for E derived in Exercise 13 of Section 1. (The motion is planar because the force is central, as in Subsection 3.1.)

The proof of Result (i) is achieved by applying the definition of the gradient and the chain rule. Thus we have

$$\operatorname{grad}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.$$

Since $\phi(\mathbf{r}) = \phi(r)$, this can be written as

$$\operatorname{grad}\phi = \frac{d\phi}{dr}\frac{\partial r}{\partial x}\mathbf{i} + \frac{d\phi}{dr}\frac{\partial r}{\partial y}\mathbf{j} + \frac{d\phi}{dr}\frac{\partial r}{\partial z}\mathbf{k} = \frac{d\phi}{dr}\left(\frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial r}{\partial z}\mathbf{k}\right).$$

But from $r = (x^2+y^2+z^2)^{1/2}$, we obtain

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{1/2} = \tfrac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2} \times 2x = \frac{x}{r}.$$

Similarly, we find that

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{and} \quad \frac{\partial r}{\partial z} = \frac{z}{r},$$

so the formula for the gradient becomes

$$\begin{aligned}\operatorname{grad}\phi &= \frac{d\phi}{dr}\left(\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}\right)\\ &= \frac{d\phi}{dr}\frac{1}{r}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{d\phi}{dr}\frac{\mathbf{r}}{r} = \frac{d\phi}{dr}\mathbf{e}_r.\end{aligned}$$

This completes the proof of Result (i).

Now watch the television programme.

TV30

4.2 Programme synopsis

Tom Smith introduces *central* forces. As an example, he demonstrates an almost frictionless puck, connected by a spring to a fixed origin but otherwise free to move on a horizontal surface. A central force acting on a particle is a force which is directed either parallel or antiparallel to its position vector $\mathbf{r}$. Such a force can be written mathematically as

$$\mathbf{F}(\mathbf{r}) = g(\mathbf{r})\mathbf{r},$$

Equation (1) of Section 3

where $g(\mathbf{r}) = g(x, y, z)$ is a function of position. It is shown that the angular momentum of the particle, $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$, is conserved (constant in time) if the particle is acted upon *only* by a central force. In this case the particle moves in a plane, which may be taken as the (x, y)-plane. With this choice of axes, the constant vector $\mathbf{L}$ is directed along the z-axis. The fact that a force is central thus allows something to be said about the corresponding particle motion, even when it is not possible to solve its equations of motion analytically.

See Subsection 3.1 for details of this analysis. It is assumed here that $\mathbf{L} \neq \mathbf{0}$. You investigated the case $\mathbf{L} = \mathbf{0}$ in Exercise 3(iii) of Section 3.

Mike Crampin takes up the story by pointing out that, since $\mathbf{L}$ is constant ($\dot{\mathbf{L}} = \mathbf{0}$), both its direction *and* magnitude are constant. The motion is in the (x, y)-plane and, using plane polar coordinates, the angular momentum is

See Exercise 4 of Section 3.

$$\mathbf{L} = L_z\mathbf{k}, \quad \text{where} \quad L_z = mr^2\dot{\theta}.$$

Since the magnitude L_z is constant, this relation can be used to eliminate either r or $\dot{\theta}$ from the equations of motion. The constancy of $r^2\dot{\theta}$ is equivalent to Kepler's second law, which says (when the force involved is that of gravity) that equal areas are swept out in equal times. So this law of Kepler, together with that part of his first law which implies that planetary orbits are planar, follows from the fact that gravity is a central force.

At the next stage, *isotropic* central forces are introduced. These are central forces whose magnitude depends only upon the distance $|\mathbf{r}|$ from the origin. Such forces have the form

$$\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r,$$

Equation (4) of Section 3

where $r = |\mathbf{r}|$ and $\mathbf{e}_r$ is the unit outward radial vector. This is a special case of the general central force, $g(\mathbf{r})\mathbf{r}$.

At this point the second result proved in Subsection 4.1 is applied. This says that if a force is isotropic as well as central then the total mechanical energy of the particle is conserved as well as its angular momentum, for then there exists a potential energy function $U(r)$ such that

$$\mathbf{F} = -\operatorname{grad} U = -\frac{dU}{dr}\mathbf{e}_r.$$

In fact, the potential energy function is

$$U(r) = -\int_{r_0}^{r} f(s)\,ds,$$

where r_0 is any non-negative constant.

Using plane polar coordinates to describe the motion, the total mechanical energy is

$$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r).$$

But $L_z = mr^2\dot{\theta}$ is constant too, so E can be written as

$$E = \tfrac{1}{2}m\dot{r}^2 + \left(\frac{L_z^2}{2mr^2} + U(r)\right).$$

The constancy of E ($\dot{E} = 0$) was shown in Subsection 1.2, and its expression in plane polar coordinates was derived in Exercise 13 of Section 1.

This equation expresses the energy solely in terms of r and $\dot{r}$, resembling the energy equation for a one-dimensional motion, so the radial motion can be considered separately. To make use of the one-dimensional analogy, it is convenient to define an effective potential energy function, U^{eff}, by

$$U^{\text{eff}}(r) = \frac{L_z^2}{2mr^2} + U(r).$$

Then the constant total mechanical energy is

$$E = \tfrac{1}{2}m\dot{r}^2 + U^{\text{eff}}(r).$$

Looking at the graph of the function U^{eff}, and using the constancy of E, allows us to see whether there are any turning points for the radial motion. At these points $\dot{r}$ vanishes, and the radial motion reverses direction. The details depend upon the precise form of the potential energy function U and on the value of E. If there are two turning points then these are the closest and furthest distances from the origin reached by the particle at the given value of energy E. If there is one turning point only then the motion is *unbound*, and the turning point is the distance of closest approach.

If the potential energy looks like that in Figure 1(a), then U^{eff} is given by the graph in Figure 1(b) and there are always two turning points, for any energy E. But for a potential energy like that of gravity, for which the graph of U is that in Figure 2(a), there will be bound or unbound motion, depending on the energy level. This is illustrated in Figure 2(b).

The potential energy given in Figure 1(a) is that for the puck attached to a perfect spring, with stiffness k and natural length r_0.

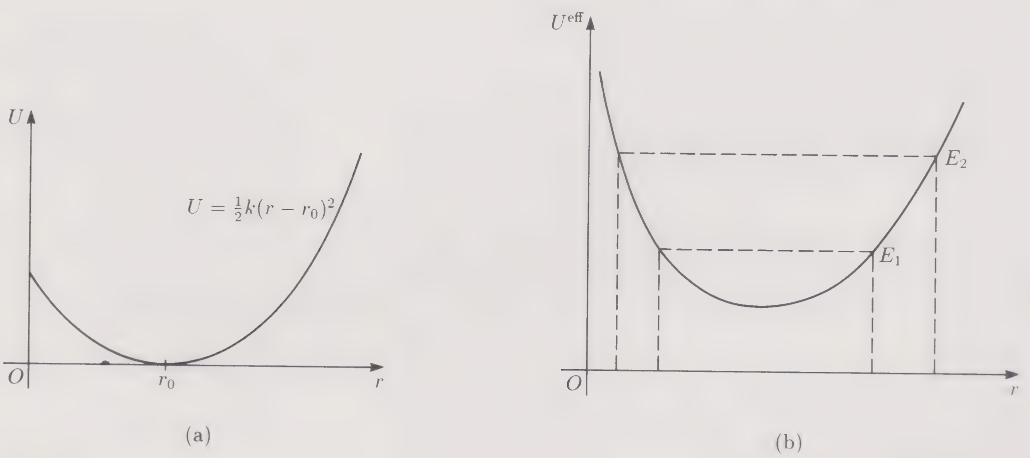


Figure 1

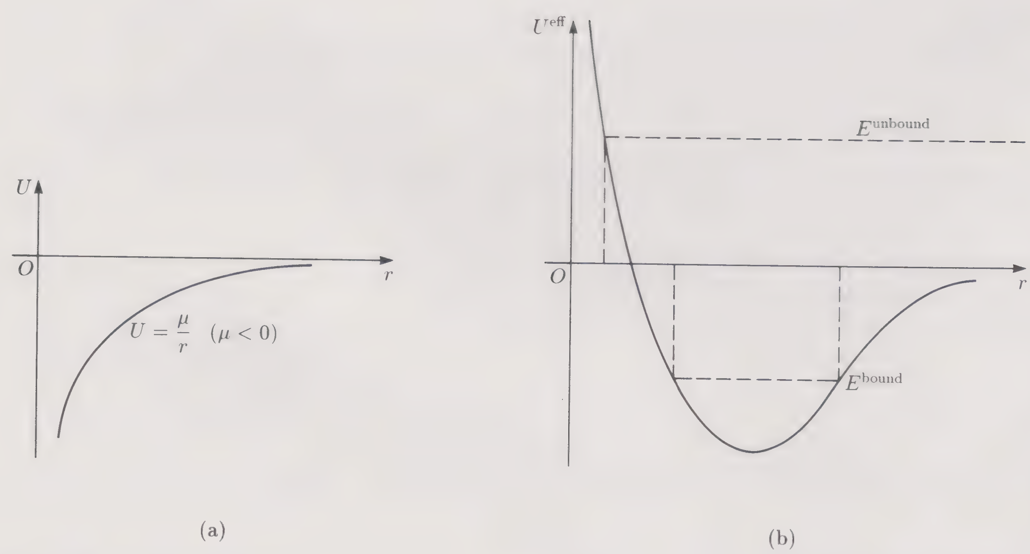


Figure 2

For gravity, the potential energy is

$$U = -\frac{GmM}{r},$$

as in Equation (19) of Section 2.

The puck connected to a spring is reconsidered in the light of this analysis. Since friction is negligible the force is central, and the effective potential energy looks like that of Figure 1(b). From this graph, inner and outer bounding radii (radial turning points) are predicted for the motion. Viewing the puck's motion from an overhead camera verifies this prediction.

In summary, isotropic central forces conserve both angular momentum and total mechanical energy. Without solving the equations of motion, the turning values for r can be found from knowledge of the potential energy function $U(r)$ and total mechanical energy E.

Two further physical examples are considered. The first is gravity, which is the main topic of this unit. The gravitational potential energy U and its corresponding effective potential energy U^{eff} are the functions shown in Figures 2(a) and 2(b) respectively. So in this case there may be two turning points, as for elliptical orbits, or only one turning point, as for unbound hyperbolic orbits.

The second example is the mutual repulsion between two electrostatic charges of equal sign, either both positive or both negative. Here the potential energy U has the same general shape as the gravitational case (Figure 2(a)), but with *opposite* sign. The functions U and U^{eff} for the electrostatic force are shown in Figures 3(a) and 3(b) below. It is clear that there are no bound states in this case. The potential U for both examples can be written in the form $U = \mu/r$, where $\mu < 0$ for gravity and $\mu > 0$ for electrostatic repulsion.

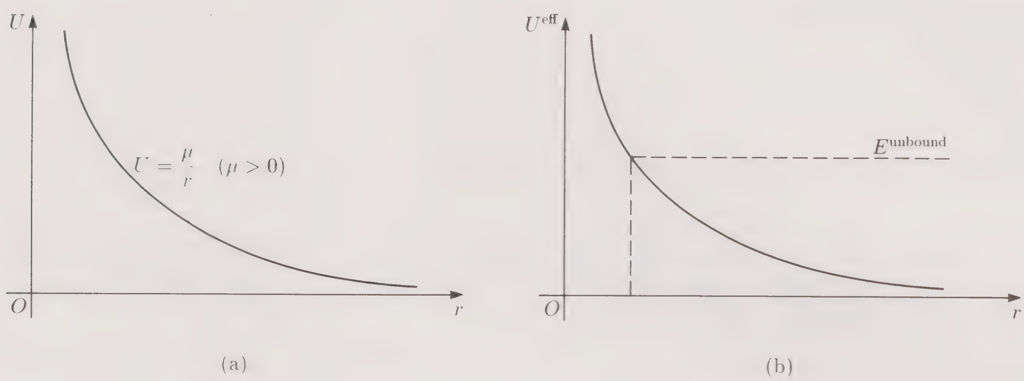


Figure 3

The physicist Ernest Rutherford carried out an experiment in the first decade of the twentieth century to demonstrate that atoms consist of relatively massive, extremely small nuclei bearing a positive electrostatic charge, surrounded by light electrons carrying a compensating negative charge. The experiment consisted of firing the lighter nuclei of helium atoms, with some energy E, at other heavier atoms and measuring the (perforce) unbound 'scattering' orbits of the helium nuclei. Rutherford achieved good agreement with his experimental data by applying the classical mechanical theory of orbits. Surprisingly, the modern theory of quantum mechanics gives similar theoretical predictions, for subtle reasons.

Summary of Section 4

1. If a scalar function $\phi(\mathbf{r})$ depends upon position only through the radial distance $|\mathbf{r}| = r = (x^2 + y^2 + z^2)^{1/2}$, so that $\phi(\mathbf{r}) = \phi(r)$, then the gradient of ϕ can be written as

$$\operatorname{grad} \phi = \frac{d\phi}{dr}\mathbf{e}_r,$$

where $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector in the outward radial direction.

2. Hence if $\mathbf{F}$ is an isotropic central force, that is,

$$\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r$$

for some function f, then $\mathbf{F}$ is conservative, with a potential energy function $U(r)$ such that

$$f(r) = -\frac{dU}{dr}.$$

The motion of a particle in this force field is planar, and has constant total mechanical energy

$$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r).$$

5 End of unit exercises

Exercise 1

For a gravitational orbit, it was shown in Example 1 of Section 3 that the total mechanical energy is

$$E = \frac{mMG}{2l}(e^2 - 1),$$

where m is the mass of the orbiting particle, M is the mass of the central body, G is the gravitational constant, and e is the eccentricity of the orbit.

(i) For a bound orbit, show that

$$E = -\frac{mMG}{2a},$$

where a is the semi-major axis of the elliptical orbit.

(ii) For any orbit, show that

$$E = \frac{mMG}{2r_{\text{min}}}(e - 1),$$

where r_{min} is the distance of closest approach of the orbiting particle to the origin.

Exercise 2

It is proposed to put a geostationary satellite into a circular orbit above the Earth's equator, such that the period of the satellite's orbit matches exactly the period of the Earth's rotation. Show that the distance d of the satellite from the Earth's centre is given by

This situation was considered previously in *Unit 28* Subsection 4.2.

$$d^3 = \frac{gR^2T^2}{4\pi^2},$$

where R is the Earth's radius and T is the period of its rotation. Taking $g = 9.81\,\text{m}\,\text{s}^{-2}$ and $R = 6.4 \times 10^6$ m, calculate the corresponding height of the satellite above the surface of the Earth.

Exercise 3

A particle of mass m is launched with speed v at right angles to the line which joins it to a spherically symmetric body of much larger mass M. At the instant of launch, the distance between m and the centre of M is R. You showed in Exercise 8 of Section 3 that, for the subsequent motion, the angular momentum has magnitude

$$L_z = mvR,$$

the total mechanical energy is

$$E = \tfrac{1}{2}mv^2 - \frac{GmM}{R},$$

and the motion is bound if

$$v^2 < \frac{2MG}{R}.$$

(i) For bound motion, show that the semi-major axis of the orbit is

$$a = \left(\frac{2}{R} - \frac{v^2}{MG}\right)^{-1}$$

(ii) For bound motion, find an expression for the eccentricity e in terms of v, R, M and G.

(iii) Show that the condition for circular motion is

$$v^2 = \frac{MG}{R}.$$

[*Solutions on page 52*]

Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

1. **(i)** Integration of $dU/dx = -k$ gives $U(x) = -kx + C$. Here C is an arbitrary constant which may be chosen equal to zero for simplicity, leaving $U(x) = -kx$.

(ii) Integration of $dU/dx = kx$ gives $U(x) = \frac{1}{2}kx^2 + C$, where C is an arbitrary constant. Choosing $C = 0$ produces $U(x) = \frac{1}{2}kx^2$.

(iii) When $U(x) = k/x$, the force is

$$F(x) = -dU/dx = k/x^2 \quad (x > 0).$$

2. Differentiation of

$$x(t) = x_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

gives

$$\dot{x}(t) = -\omega x_0 \sin\omega t + v_0 \cos\omega t.$$

(Note here that $x(0) = x_0$ and $\dot{x}(0) = v_0$, as claimed.)

Squaring each of these equations gives

$$x^2 = x_0^2\cos^2\omega t + \frac{2x_0v_0}{\omega}\cos\omega t\sin\omega t + \frac{v_0^2}{\omega^2}\sin^2\omega t$$

and

$$\dot{x}^2 = \omega^2x_0^2\sin^2\omega t - 2\omega x_0v_0\cos\omega t\sin\omega t + v_0^2\cos^2\omega t.$$

The total mechanical energy is

$$E(t) = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 + C$$
$$= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\omega^2x^2 + C.$$

On substituting for x^2 and $\dot{x}^2$ from above and simplifying, this becomes

$$E(t) = \tfrac{1}{2}m\omega^2x_0^2 + \tfrac{1}{2}mv_0^2 + C = E(0).$$

Hence, by direct substitution, it has been shown that E is constant.

At various times the particle reaches its turning points, located at x_T say. Since the speed vanishes at these points, the energy there can be expressed as

$$E = \tfrac{1}{2}kx_T^2 + C = \tfrac{1}{2}m\omega^2x_T^2 + C.$$

Hence, since E is constant throughout the motion, we have

$$\tfrac{1}{2}m\omega^2x_T^2 + C = \tfrac{1}{2}m\omega^2x_0^2 + \tfrac{1}{2}mv_0^2 + C,$$

from which the locations of the turning points are deduced to be

$$x_T = \pm\sqrt{x_0^2 + v_0^2/\omega^2}.$$

3. Since the particle is released from rest, its initial velocity $\dot{x}(0)$ is zero, so that its total mechanical energy is then

$$E = \tfrac{1}{2}m\dot{x}^2(0) + U(\alpha) = U(\alpha).$$

The energy is constant throughout. At $x = \beta$ it can be written as

$$E = \tfrac{1}{2}mv^2(\beta) + U(\beta),$$

where $v(x) = \dot{x}$. Equating the two expressions for E gives

$$\tfrac{1}{2}mv^2(\beta) + U(\beta) = U(\alpha),$$

so the speed at $x = \beta$ is

$$v(\beta) = \sqrt{\frac{2}{m}\left(U(\alpha) - U(\beta)\right)}.$$

Similarly, the speed at $x = \gamma$ is

$$v(\gamma) = \sqrt{\frac{2}{m}\left(U(\alpha) - U(\gamma)\right)}.$$

For $x \geq \delta$ the potential energy function $U(x)$ is zero, so the speed is then constant, with value

$$v = \sqrt{\frac{2U(\alpha)}{m}}.$$

From Figure 6 of Section 1, or by considering these formulas for the speed and remembering that $U(\beta) < 0, U(\gamma) > 0$, we have

$$v(\beta) > v(\delta) > v(\gamma).$$

4. The particle is projected from a point to the right of $x = \delta$, where the potential energy vanishes (see Figure 6 of Section 1). Since its initial velocity is $\dot{x}(0)$, the energy is

$$E = \tfrac{1}{2}m\dot{x}^2(0),$$

from which we have

$$\dot{x}(0) = -\sqrt{\frac{2E}{m}}.$$

The negative sign here arises because the particle is projected to the left. After projection it slows down to a minimum speed at $x = \gamma$ and then accelerates to an overall maximum speed at $x = \beta$. After reaching this point it slows to a halt at $x = \alpha$. Thereafter it retraces its path, with a maximum speed at $x = \beta$ and minimum speed at $x = \gamma$. After reaching $x = \delta$ it moves out to infinity with velocity equal to its initial speed, that is,

$$\dot{x} = \sqrt{\frac{2E}{m}}.$$

5. Using the definition of the gradient, the vector equation

$$\mathbf{F} = -\operatorname{grad} U,$$

with $\mathbf{F} = -mg\mathbf{j}$, is equivalent to the three scalar equations

$$0 = -\frac{\partial U}{\partial x}, \quad -mg = -\frac{\partial U}{\partial y}, \quad 0 = -\frac{\partial U}{\partial z}.$$

By inspection, these equations have the solution

$$U = mgy + C,$$

where C is any constant. Thus the uniform gravitational field is conservative, with the potential energy function just found.

6. We seek

$$\mathbf{F} = -\operatorname{grad} U = -\frac{\partial U}{\partial x}\mathbf{i} - \frac{\partial U}{\partial y}\mathbf{j} - \frac{\partial U}{\partial z}\mathbf{k},$$

where

$$U = k/r = k(x^2 + y^2 + z^2)^{-1/2}.$$

Now we have

$$\frac{\partial U}{\partial x} = -\tfrac{1}{2}k(x^2 + y^2 + z^2)^{-3/2} \times 2x = -kxr^{-3}.$$

Similarly, it can be shown that

$$\frac{\partial U}{\partial y} = -kyr^{-3} \quad\text{and}\quad \frac{\partial U}{\partial z} = -kzr^{-3},$$

so that

$$\mathbf{F} = \frac{k}{r^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{k\mathbf{r}}{r^3}.$$

7. It is given that

$$m\ddot{\mathbf{r}} = -\operatorname{grad} U + \mathbf{G} \quad\text{and}\quad \mathbf{G}\cdot\dot{\mathbf{r}} = 0. \tag{1}$$

Differentiating $E = \frac{1}{2}m\dot{\mathbf{r}}^2 + U$ and applying the chain rule leads, as in the text, to the equation

$$\dot{E} = \dot{\mathbf{r}}\cdot(m\ddot{\mathbf{r}} + \operatorname{grad} U).$$

Using each of Equations (1) in turn, this becomes

$$\dot{E} = \dot{\mathbf{r}}\cdot\mathbf{G} = 0.$$

8. As explained in the text prior to the exercise, the total mechanical energy E is constant, where

$$E = \tfrac{1}{2}mu^2 + mgy + C.$$

Initially, the particle's speed is zero, since it is released from rest, and its height is $y = h$, so that $E = mgh + C$. When it reaches the origin we have $y = 0$ and $u = u_0$, so the energy can also be written as $E = \tfrac{1}{2}mu_0^2 + C$. Hence

$$\tfrac{1}{2}mu_0^2 + C = mgh + C,$$

giving the particle's speed at the origin as

$$u_0 = \sqrt{2gh}.$$

(Note that this outcome is independent of the slope of the plane.)

9. Since $\mathbf{i}, \mathbf{j}$ form an orthogonal pair of unit vectors, we have

$$\mathbf{i} \cdot \mathbf{j} = 0 \quad \text{and} \quad \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1.$$

Then Equations (9) of Section 1 give

$$\begin{aligned}\mathbf{e}_r \cdot \mathbf{e}_\theta &= (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \cdot (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) \\ &= -\cos\theta\sin\theta\,\mathbf{i}\cdot\mathbf{i} + (\cos^2\theta - \sin^2\theta)\mathbf{i}\cdot\mathbf{j} \\ &\quad + \sin\theta\cos\theta\,\mathbf{j}\cdot\mathbf{j} \\ &= 0.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\mathbf{e}_r \cdot \mathbf{e}_r &= (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \cdot (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \\ &= \cos^2\theta + \sin^2\theta = 1\end{aligned}$$

and

$$\begin{aligned}\mathbf{e}_\theta \cdot \mathbf{e}_\theta &= (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) \cdot (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) \\ &= \sin^2\theta + \cos^2\theta = 1.\end{aligned}$$

10. Since the particle is constrained to move at a fixed radius $r = R$, we have $\dot{r} = \ddot{r} = 0$. Then, in plane polar coordinates,

$$\dot{\mathbf{r}} = R\dot{\theta}\mathbf{e}_\theta \quad \text{and} \quad \ddot{\mathbf{r}} = -R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta.$$

(These are the results which you obtained in Exercise 7 of *Unit 28* Section 4, and which were derived again in Subsection 4.2 of that unit.) Using Equations (9) of Section 1 to substitute for $\mathbf{e}_r, \mathbf{e}_\theta$ in terms of $\mathbf{i}, \mathbf{j}$ gives

$$\dot{\mathbf{r}} = -R\dot{\theta}\sin\theta\,\mathbf{i} + R\dot{\theta}\cos\theta\,\mathbf{j},$$

$$\ddot{\mathbf{r}} = -R\left(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right)\mathbf{i} + R\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\mathbf{j}.$$

11. The polar angle θ is fixed here at the value θ_0. Hence $\dot{\theta} = \ddot{\theta} = 0$. Then, in plane polar coordinates,

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r \quad \text{and} \quad \ddot{\mathbf{r}} = \ddot{r}\mathbf{e}_r.$$

In terms of the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$, these are

$$\dot{\mathbf{r}} = \dot{r}\cos\theta_0\,\mathbf{i} + \dot{r}\sin\theta_0\,\mathbf{j},$$

$$\ddot{\mathbf{r}} = \ddot{r}\cos\theta_0\,\mathbf{i} + \ddot{r}\sin\theta_0\,\mathbf{j}.$$

12. We have

$$r = at^2 + bt, \quad \dot{r} = 2at + b, \quad \ddot{r} = 2a,$$

and $\quad \theta = ct^2, \quad \dot{\theta} = 2ct, \quad \ddot{\theta} = 2c,$

so that the velocity and acceleration are given by

$$\dot{\mathbf{r}} = (2at + b)\mathbf{e}_r + (at^2 + bt)(2ct)\mathbf{e}_\theta$$

and

$$\begin{aligned}\ddot{\mathbf{r}} &= [2a - (at^2 + bt)(2ct)^2]\mathbf{e}_r \\ &\quad + [2(2at + b)(2ct) + (at^2 + bt)(2c)]\mathbf{e}_\theta.\end{aligned}$$

13. The velocity of the particle is

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta,$$

and since $\mathbf{e}_r, \mathbf{e}_\theta$ are orthogonal unit vectors (with $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$ and $\mathbf{e}_r \cdot \mathbf{e}_r = \mathbf{e}_\theta \cdot \mathbf{e}_\theta = 1$) the total mechanical energy is

$$\begin{aligned}E &= \tfrac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + U \\ &= \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U.\end{aligned}$$

14. (i) The force acting on the particle is $\mathbf{F} = -T\mathbf{e}_r$. Since r is a linear function of t, we have $\ddot{r} = 0$. Hence Newton's second law, in the form of Equations (14) of Section 1, gives

$$-mr\dot{\theta}^2 = -T \quad \text{and} \quad \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0.$$

These equations are equivalent to

$$mr\dot{\theta}^2 = T, \tag{2}$$

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0. \tag{3}$$

(ii) From Equation (3), we know that $r^2\dot{\theta}$ is constant. Hence we have

$$r^2(t)\dot{\theta}(t) = r^2(0)\dot{\theta}(0).$$

This is used to substitute for $\dot{\theta}(t)$ in Equation (2), giving

$$T = mr(t)\left(\frac{r^2(0)\dot{\theta}(0)}{r^2(t)}\right)^2 = \frac{mr^4(0)\dot{\theta}^2(0)}{r^3(t)} = \frac{mr^4(0)\dot{\theta}^2(0)}{(r(0) - \mu t)^3}$$

as required.

(iii) The string will break when $T = 500\,\mathrm{N}$. From the answer to part (ii), this occurs when

$$\frac{mr^4(0)\dot{\theta}^2(0)}{r^3} = 500,$$

or when

$$r = \left(mr^4(0)\dot{\theta}^2(0)/500\right)^{1/3}.$$

With $m = 0.1\,\mathrm{kg}$, $r(0) = 1\,\mathrm{m}$ and $\dot{\theta}(0) = 1\,\mathrm{rad\,s^{-1}}$, this gives $r = (0.1/500)^{1/3} \simeq 0.058\,\mathrm{m}$. The time at which the string breaks is then

$$t = (r(0) - r)/\mu = 60(1 - r) \simeq 56.5\,\mathrm{s}.$$

(Note, from the form of the function for T, that for any finite breaking strength the string will break before the particle reaches the origin.)

15. (i) The force is $-k(r - l_0)\mathbf{e}_r$ so that, from Equations (14) of Section 1, the equations of motion are

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -k(r - l_0),$$

$$m\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0.$$

(ii) From the last equation, $mr^2\dot{\theta}$ must be constant. (This quantity is the *angular momentum* of the particle. Conservation of angular momentum occurred also in the previous exercise, and will be of great importance in the derivation of planetary orbits.)

(iii) If $U = \tfrac{1}{2}k(r - l_0)^2$, where $r = (x^2 + y^2)^{1/2}$, then we have

$$\begin{aligned}\frac{\partial U}{\partial x} &= \tfrac{1}{2}k \times 2(r - l_0)\frac{\partial r}{\partial x} \\ &= k(r - l_0) \times \tfrac{1}{2}(x^2 + y^2)^{-1/2} \times 2x \\ &= k(r - l_0)xr^{-1}.\end{aligned}$$

Similarly,

$$\frac{\partial U}{\partial y} = k(r - l_0)yr^{-1},$$

so that (using Equation (8) of Section 1)

$$\begin{aligned}-\operatorname{grad} U &= -\frac{\partial U}{\partial x}\mathbf{i} - \frac{\partial U}{\partial y}\mathbf{j} \\ &= -k(r - l_0)r^{-1}(x\mathbf{i} + y\mathbf{j}) \\ &= -k(r - l_0)r^{-1}\mathbf{r} \\ &= -k(r - l_0)\mathbf{e}_r.\end{aligned}$$

Since this expression for $-\operatorname{grad} U$ is equal to the force $\mathbf{F}$ given in the statement of the exercise, we have shown that U is a potential energy function for $\mathbf{F}$ (and hence established that $\mathbf{F}$ is a conservative force).

(iv) Suppose that the speed of the particle is denoted by v. From part (iii), the total mechanical energy

$$E = \tfrac{1}{2}mv^2 + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}k(r - l_0)^2$$

is constant. When $r = 2l_0$, the speed is given as $v = u$, so that

$$E = \tfrac{1}{2}mu^2 + \tfrac{1}{2}kl_0^2.$$

When $r = l_0$ the potential energy is zero, so that the speed v is then given by $E = \frac{1}{2}mv^2$. Hence we have

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mu^2 + \tfrac{1}{2}kl_0^2$$

or $v = \sqrt{u^2 + \dfrac{kl_0^2}{m}}$

for the speed of the puck when the spring has its natural length.

Solutions to the exercises in Section 2

1. When $a = b$, the equation of the ellipse can be written as

$$X^2 + Y^2 = a^2.$$

This is the equation of a circle with radius a.

2. (i) The area A is twice the area of the upper half of the ellipse, namely

$$A = 2\int_{-a}^{a} Y(X)\,dX.$$

But from the definition of the ellipse (Equation (1) of Section 2), the upper half of the ellipse (where $Y > 0$) has equation $Y = b\sqrt{1-(X/a)^2}$. Thus the area is

$$A = 2b\int_{-a}^{a} \sqrt{1-(X/a)^2}\,dX.$$

(ii) Let $X = a\cos u$. When $X = -a$, $u = \pi$, when $X = a$, $u = 0$, and $dX = -a\sin u\,du$, so that the integral becomes

$$A = -2ab\int_{\pi}^{0} \sin^2 u\,du = 2ab\int_{0}^{\pi} \sin^2 u\,du$$
$$= ab\int_{0}^{\pi}(1-\cos 2u)\,du = \pi ab.$$

3. A circle corresponds to $a = b$, for which $q = 0$. Hence, for a circle, the foci coincide at the centre.

4. (i) When $b = a$ the eccentricity is $e = 0$.

(ii) When $b = \frac{1}{2}a$, we have $e = \frac{1}{2}\sqrt{3} \simeq 0.866$.

(iii) When $b = \frac{1}{3}a$, we have $e = \frac{1}{3}\sqrt{8} \simeq 0.943$.

The larger the semi-major axis a is compared to the semi-minor axis b, the more 'squashed' is the shape of the ellipse. This corresponds to larger values of the eccentricity e.

5. From Equation (9) of Section 2, we have

$$\dot{A} = \tfrac{1}{2}R^2\dot{\theta}.$$

But also $V = R\dot{\theta}$, so that

$$\dot{A} = \tfrac{1}{2}RV.$$

Since both R and V are constants, $\dot{A}$ is constant.

6. Refer to Figure 4 of Section 2. During the interval $t_1 \le t \le t_2$, the planet is *closer* to the origin than it is during the interval $t_3 \le t \le t_4$. So to sweep through the same area (as it must do by Kepler's second law), it moves *faster* during the first interval.

7. If T is the planet's period and a is its semi-major axis, then Kepler's third law states that

$$T^2 = ka^3,$$

where k is a constant. If $T_{\rm M}, a_{\rm M}$ are respectively the period and semi-major axis for Mars, and $T_{\rm E}, a_{\rm E}$ are the corresponding parameters for the Earth, then we have

$$\left(\frac{T_{\rm M}}{T_{\rm E}}\right)^2 = \left(\frac{a_{\rm M}}{a_{\rm E}}\right)^3 \quad\text{or}\quad T_{\rm M} = T_{\rm E}\left(\frac{a_{\rm M}}{a_{\rm E}}\right)^{3/2}.$$

For the values given,

$$T_{\rm M} = 365.256 \times (1.524)^{3/2} \simeq 687.187 \text{ days}.$$

8. In terms of units, Equation (10) of Section 2 for the force of gravity gives

$$\text{force} = \frac{G \times (\text{mass})^2}{(\text{length})^2},$$

where '=' here means 'has the same units as'. It follows that

$$G = \frac{(\text{length})^2 \times \text{force}}{(\text{mass})^2}.$$

But from Newton's second law, $F = ma$, we have

$$\text{force} = \frac{\text{mass} \times \text{length}}{(\text{time})^2},$$

so that

$$G = \frac{(\text{length})^2}{(\text{mass})^2} \times \frac{\text{mass} \times \text{length}}{(\text{time})^2} = \frac{(\text{length})^3}{\text{mass} \times (\text{time})^2}.$$

Thus, in SI units, G has the units $\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$.

9. The given vector is of the form $\mathbf{a}/|\mathbf{a}|$, where $\mathbf{a} = \mathbf{r}_j - \mathbf{r}_i$. Any vector of this form is a unit vector, since its magnitude is $|\mathbf{a}|/|\mathbf{a}| = 1$.

10. Equation (10) of Section 2 gives

$$\mathbf{F}_{ij} = \frac{Gm_im_j}{|\mathbf{r}_j - \mathbf{r}_i|^3}(\mathbf{r}_j - \mathbf{r}_i)$$

as the force on m_i due to m_j. Hence, swapping the indices i and j, the force on m_j due to m_i is

$$\mathbf{F}_{ji} = \frac{Gm_jm_i}{|\mathbf{r}_i - \mathbf{r}_j|^3}(\mathbf{r}_i - \mathbf{r}_j) = -\frac{Gm_im_j}{|\mathbf{r}_j - \mathbf{r}_i|^3}(\mathbf{r}_j - \mathbf{r}_i),$$

where we have used the fact that $|\mathbf{r}_i - \mathbf{r}_j| = |\mathbf{r}_j - \mathbf{r}_i|$. Thus $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, and Newton's third law is satisfied by the gravitational force.

11. Equation (15) of Section 2 gives the gravitational potential energy function resulting from a system of particles $m_1, m_2, \ldots, m_N$, acting upon another particle of mass m located at $\mathbf{r}$. Here there is only a single particle in the system ($N = 1$), with mass $m_1 = M$ and position vector $\mathbf{r}_1 = \mathbf{0}$. Hence the potential energy function is

$$U(\mathbf{r}) = -\frac{GmM}{|\mathbf{r}|} = -\frac{GmM}{r}.$$

The corresponding gravitational force is

$$\mathbf{F}(\mathbf{r}) = -\operatorname{grad} U(\mathbf{r}).$$

From Equation (13) of Section 2, with $\mathbf{r}_i = \mathbf{r}_1 = \mathbf{0}$, this gives

$$\mathbf{F}(\mathbf{r}) = GmM \operatorname{grad}\frac{1}{|\mathbf{r}|} = -\frac{GmM\mathbf{r}}{|\mathbf{r}|^3},$$

which could also have been obtained directly from Equation (12) of Section 2. Putting $\mathbf{e}_r = \mathbf{r}/|\mathbf{r}|$ for the unit vector in the radially outward direction, this can be written as

$$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{r^2}\mathbf{e}_r.$$

12. By definition, a spherically symmetric mass distribution has a mass density function ρ which depends on $\mathbf{r}$ only through its magnitude

$$|\mathbf{r}| = r = \sqrt{x^2 + y^2 + z^2}.$$

Of the functions listed, only (c), (d) and (f) meet this criterion.

13. From Equation (19) of Section 2, the gravitational potential energy of the particle of mass m is

$$U(\mathbf{r}) = -\frac{GmM}{r}.$$

This is the same potential energy as was found in Exercise 11 for gravitational attraction by a single particle of mass M at the origin. Hence the gravitational force on the particle of mass m is, as before,

$$\mathbf{F}(\mathbf{r}) = -\operatorname{grad} U(\mathbf{r}) = -\frac{GmM}{r^2}\mathbf{e}_r.$$

14. Suppose that Body 1 has mass M and that Body 2 has mass m. Using the given hint, the motion of the centre of mass C_2 of Body 2 (which is also its geometric centre, by spherical symmetry) is the same as that of a particle of mass m acted upon by the same total force. However, we know from the result of Exercise 13 that the gravitational force experienced by such a particle due to Body 1 is directed from C_2 to the centre C_1 of Body 1, with magnitude GmM/R^2. This is therefore the total gravitational force exerted by Body 1 upon Body 2.

By a similar argument, or by application of Newton's third law, Body 2 exerts upon Body 1 a force of the same magnitude but in the opposite direction, that is, from C_1 to C_2.

15. (i) The magnitude of the gravitational force on a particle of mass m at the Earth's surface (where $r = R$) is mg (from the simple model) and mMG/R^2 (from Newton's universal law of gravitation). Since these must be equal, we have $g = MG/R^2$.

(ii) From the given values, we obtain

$$M = \frac{gR^2}{G} = \frac{9.81 \times (6.4 \times 10^6)^2}{6.67 \times 10^{-11}} \simeq 6.0 \times 10^{24}\,\text{kg}.$$

16. At height h above the surface, the gravitational force on a particle of mass m has magnitude $mMG/(R+h)^2$, so that the gravitational acceleration at this height is

$$\frac{MG}{(R+h)^2}.$$

At ground level, we have

$$g = \frac{MG}{R^2}$$

from Exercise 15(i). The gravitational acceleration at height h is therefore

$$\frac{gR^2}{(R+h)^2} = \frac{g}{(1+h/R)^2}.$$

With $g = 9.81\,\text{m}\,\text{s}^{-2}$, $h = 10^4$ m and $R = 6.4 \times 10^6$ m, this gives the value $9.78\,\text{m}\,\text{s}^{-2}$ for the gravitational acceleration at a height of 10 km.

Solutions to the exercises in Section 3

1. Forces (a), (b), (c) and (f) are central forces, since these are directed along $\mathbf{r}$ or along $-\mathbf{r}$, depending on the sign of the constant K (and, in the case of (c), on the sign of x).

2. From the result of Exercise 13 in Section 2, the gravitational force acting on the particle of mass m is

$$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{r^3}\mathbf{r},$$

giving $g(\mathbf{r}) = -GmM/r^3$.

3. (i) By the definition of the cross product of two vectors (*Unit 14* Subsection 3.5),

$$\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$$

is perpendicular to $\mathbf{r}$ whenever $\mathbf{L} \neq \mathbf{0}$.

(ii) Combining Newton's second law and the definition of a central force (Equation (1) of Section 3), we have

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}) = g(\mathbf{r})\mathbf{r},$$

for some scalar function $g(\mathbf{r})$. Then we obtain

$$\begin{aligned}\dot{\mathbf{L}} &= \frac{d}{dt}(\mathbf{r} \times m\dot{\mathbf{r}})\\ &= \dot{\mathbf{r}} \times m\dot{\mathbf{r}} + \mathbf{r} \times m\ddot{\mathbf{r}}\\ &= \mathbf{0} + \mathbf{r} \times g(\mathbf{r})\mathbf{r} = \mathbf{0},\end{aligned}$$

where we have twice used the fact that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$. Hence $\mathbf{L}$ is a constant vector.

(iii) Referring once more to the definition of the cross product, if $\mathbf{L} = \mathbf{0}$ and $\mathbf{r} \neq \mathbf{0}, \dot{\mathbf{r}} \neq \mathbf{0}$, then $\dot{\mathbf{r}}$ must be in the direction of $\mathbf{r}$ or of $-\mathbf{r}$.

4. (i) The angular momentum of the particle is

$$\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}},$$

but since the motion is in the (x, y)-plane, we may apply the equations

$$\mathbf{r} = r\mathbf{e}_r \quad \text{and} \quad \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

(Equations (8) and (12) of Section 1). Since $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{k}$, we obtain

$$\begin{aligned}\mathbf{L} &= r\mathbf{e}_r \times m\left(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta\right)\\ &= mr\dot{r}\mathbf{e}_r \times \mathbf{e}_r + mr^2\dot{\theta}\mathbf{e}_r \times \mathbf{e}_\theta = mr^2\dot{\theta}\mathbf{k}.\end{aligned}$$

Hence $\mathbf{L} = L_z\mathbf{k}$, where $L_z = mr^2\dot{\theta}$.

(ii) Since m is constant, $\dot{L}_z = 0$ is equivalent to

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0,$$

which is Equation (3) of Section 3.

5. (i) Isotropic central forces have the form $\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r$. Of the forces listed in Exercise 1, the following are isotropic central forces:

(a) since $\mathbf{F}(\mathbf{r}) = Kr\mathbf{e}_r$;
(b) since $\mathbf{F}(\mathbf{r}) = (K/r)\mathbf{e}_r$;
(f) since $\mathbf{F}(\mathbf{r}) = K\mathbf{e}_r$.

(ii) For the gravitational force we have, from Exercise 2 (or from Exercise 13 in Section 2),

$$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{r^3}\mathbf{r} = -\frac{GmM}{r^2}\mathbf{e}_r,$$

so this is an isotropic central force with $f(r) = -GmM/r^2$.

6. The effective potential energy function is

$$U^{\text{eff}}(r) = \frac{L_z^2}{2mr^2} - \frac{GmM}{r}.$$

Differentiation of this equation gives

$$\frac{dU^{\text{eff}}}{dr} = -\frac{L_z^2}{mr^3} + \frac{GmM}{r^2}.$$

Now $dU^{\text{eff}}/dr = 0$ at $r = r^*$, which leads to the solution

$$r^* = \frac{L_z^2}{m^2MG}.$$

The corresponding effective potential energy is

$$U^{\text{eff}}_{\text{min}} = U^{\text{eff}}(r^*) = \frac{L_z^2}{2m(r^*)^2} - \frac{GmM}{r^*} = -\frac{m^3M^2G^2}{2L_z^2}.$$

7. If $E = U^{\text{eff}}_{\text{min}}$ then we have $\dot{r} = 0$ and $r = r^*$ at all times. This orbit is therefore a circle of radius r^*.

8. (i) The angular momentum is $L_z = mr^2\dot{\theta}$. At the instant of launch we have $r = R$ and, since there is then no radial component of velocity, the initial speed is $v = R\dot{\theta}$. Hence $L_z = mvR$.

(ii) The total mechanical energy E is the sum of the kinetic and potential energies, and is constant. Initially, the kinetic energy is $\frac{1}{2}mv^2$ and the potential energy is $-GmM/R$. Hence, for all subsequent times,

$$E = \tfrac{1}{2}mv^2 - \frac{GmM}{R}.$$

(iii) The motion is bound provided that $E < 0$. So for bound motion, we must have

$$\tfrac{1}{2}mv^2 - \frac{GmM}{R} < 0, \quad \text{or} \quad v^2 < \frac{2MG}{R}.$$

9. (i) Since $v^2 = MG/(2R)$ the total mechanical energy is, from Exercise 8(ii),

$$E = \tfrac{1}{2}mv^2 - \frac{GmM}{R} = -\frac{3GmM}{4R}.$$

Since E is negative, the motion is bound.

(ii) In general, bound motion has two turning points (although for the special case of motion in a circle the radius, r, is constant). From the given hint, the equation for turning points is $E = U^{\text{eff}}(r)$, where

$$U^{\text{eff}}(r) = \frac{L_z^2}{2mr^2} - \frac{GmM}{r}.$$

From Exercise 8(i), we have $L_z = mvR$, so the equation for r_{min} and r_{max} is

$$\frac{(mvR)^2}{2mr^2} - \frac{GmM}{r} = E = -\frac{3GmM}{4R}.$$

Putting $s = R/r$, the equation becomes

$$\frac{mv^2s^2}{2} - \frac{GmMs}{R} = -\frac{3GmM}{4R}.$$

The given condition $v^2 = MG/(2R)$ simplifies this to

$$\tfrac{1}{4}s^2 - s = -\tfrac{3}{4}, \quad \text{or} \quad s^2 - 4s + 3 = 0.$$

The solutions are $s = 1$ and $s = 3$, which correspond to $r_{\text{max}} = R$ and $r_{\text{min}} = \frac{1}{3}R$.

(We must assume here that the spherically symmetric body at the origin has a radius less than $\frac{1}{3}R$. Otherwise the orbiting particle would collide with it!)

10. (i) The equation of orbits is

$$\frac{l}{r} = 1 + e\cos(\theta - \theta_0). \tag{1}$$

The right-hand side of this equation takes values between $1 - e$ and $1 + e$ (inclusive) as θ varies. The minimum value of r corresponds to the maximum value of the right-hand side, which is $1 + e$. Hence we have

$$r_{\text{min}} = \frac{l}{1+e}.$$

(ii) If $e < 1$ then $1 - e > 0$, so that the right-hand side of Equation (1) is always positive. Its minimum value $1 - e$ corresponds to the maximum value of r, giving

$$r_{\text{max}} = \frac{l}{1-e}.$$

(iii) If $e \geq 1$ then $1 - e \leq 0$ and the treatment of part (ii) cannot apply, since r is always non-negative. The condition $r > 0$ is equivalent to

$$1 + e\cos(\theta - \theta_0) > 0,$$

which is satisfied by the range of angles

$$\theta_0 - \theta_a < \theta < \theta_0 + \theta_a,$$

where $\theta_a = \arccos(-e^{-1})$. For this range of values of θ there is no upper bound for the values of r, so the motion is unbound in this case. The corresponding orbits (for $e > 1$) are hyperbolas, as illustrated in the diagram below.

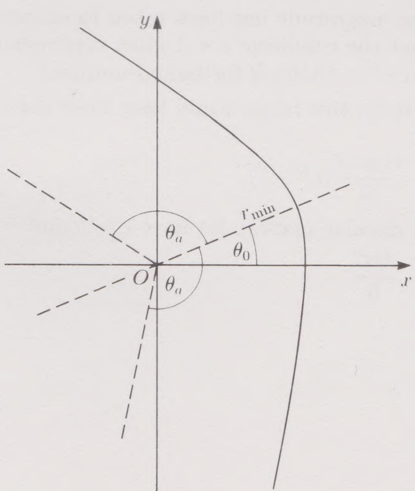


Figure 1

(iv) Orbits are bound for $e < 1$ and unbound for $e \geq 1$.

11. When $e = 0$ we have (from Equations (18) and (16) of Section 3)

$$r = l = \frac{L_z^2}{m^2MG}.$$

This is a circle of radius l, for which $r_{\text{max}} = r_{\text{min}} = l$. From Example 1 of Section 3, the total mechanical energy is

$$E = -\frac{mMG}{2l} = -\frac{m^3M^2G^2}{2L_z^2}.$$

(The radius of the circle agrees with the value of r^* found in Exercise 6, as expected from Exercise 7. Since $\dot{r} = 0$, the corresponding energy is $E = U^{\text{eff}}(r^*)$, and the value of E just obtained agrees with that for $U^{\text{eff}}(r^*)$ found in Exercise 6.)

12. (i) From Equation (13) of Section 1, the acceleration is

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta.$$

Here $r = d$ and $\dot{\theta}$ are constants, so that $\dot{r} = \ddot{r} = \ddot{\theta} = 0$. Hence

$$\ddot{\mathbf{r}} = -d\dot{\theta}^2\mathbf{e}_r,$$

showing that the acceleration is in the inward radial direction, with magnitude $|\ddot{\mathbf{r}}| = d\dot{\theta}^2$.

(ii) From Equations (8) and (12) of Section 1, we have

$$\mathbf{r} = r\mathbf{e}_r \quad \text{and} \quad \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta = r\dot{\theta}\mathbf{e}_\theta,$$

so that the angular momentum is

$$\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}} = mr^2\dot{\theta}\mathbf{e}_r \times \mathbf{e}_\theta = md^2\dot{\theta}\mathbf{k}.$$

Thus the magnitude of the angular momentum is $L_z = md^2\dot{\theta}$.

(iii) The gravitational force on the satellite is (from Exercise 13 of Section 2)

$$\mathbf{F} = -\frac{GmM}{r^2}\mathbf{e}_r = -\frac{GmM}{d^2}\mathbf{e}_r,$$

so that, from part (i), $m\ddot{\mathbf{r}} = \mathbf{F}$ gives

$$d\dot{\theta}^2 = \frac{MG}{d^2}. \tag{2}$$

Using the result of part (ii) to eliminate $\dot{\theta}$, this becomes

$$\frac{L_z^2}{m^2d^3} = \frac{MG}{d^2} \quad \text{or} \quad d = \frac{L_z^2}{m^2MG},$$

in agreement with the value of the orbital radius l found in Exercise 11.

(iv) The gravitational potential energy is $-GmM/d$. The kinetic energy is (from Equation (2))

$$\tfrac{1}{2}m(d\dot{\theta})^2 = \frac{GmM}{2d},$$

so that the sum of the kinetic and potential energies is

$$E = -\frac{GmM}{2d}.$$

With $d = l$, this is the same as the expression for E found in Exercise 11.

13. (i) If the Earth is assumed to be spherically symmetric, and d is the distance between the Moon's centre of mass and the Earth's centre, then by Newton's universal law of gravitation, the magnitude of the gravitational force exerted on the Moon is $|\mathbf{F}| = GmM/d^2$.

(ii) We have $|\ddot{\mathbf{r}}| = d\dot{\theta}^2$. The angular velocity $\dot{\theta}$ is constant, and therefore equals $2\pi/T$, where T is the orbital period. It follows that

$$|\ddot{\mathbf{r}}| = \frac{4\pi^2d}{T^2}.$$

(iii) The result $g = MG/R^2$ is shown as in the solution to Exercise 15(i) of Section 2.

(iv) From Newton's second law $m\ddot{\mathbf{r}} = \mathbf{F}$ applied to the gravitational motion of the Moon, and from parts (i) and (ii), we have

$$\frac{4\pi^2 dm}{T^2} = \frac{GmM}{d^2},$$

which is equivalent to

$$T^2 = \left(\frac{4\pi^2}{GM}\right) d^3.$$

From part (iii), this is the same as

$$T^2 = \left(\frac{4\pi^2}{gR^2}\right) d^3,$$

as required.

(v) Using $d = 60R$, $g = 9.8\,\mathrm{m\,s^{-2}}$, $T = 28 \times 24 \times 3600\,\mathrm{s}$ and $R = 5.3 \times 10^6$ m gives

$$\frac{4\pi^2 d^3}{gR^2T^2} = \frac{4\pi^2 \times 60^3 \times 5.3 \times 10^6}{9.8 \times 28^2 \times 24^2 \times 3600^2} \simeq 0.8.$$

This is the ratio of the two sides of the equation in part (iv), which should be close to 1 if the prediction is accurate. (If $R = 6.4 \times 10^6$ m is used instead of 5.3×10^6 m, with the other values unchanged, then the value of this ratio becomes 0.95.)

14. (i) In a time T the whole area of the ellipse is swept out. Since $\dot{A}$ is constant, we then have

$$\pi ab = \dot{A}T = \frac{L_zT}{2m}.$$

Squaring both sides of this equation, and putting

$$b^2 = al = \frac{aL_z^2}{m^2MG},$$

we obtain

$$T^2 = \left(\frac{4\pi^2}{MG}\right) a^3,$$

as required.

(ii) For the circular orbit of Exercise 13, the orbital radius is $a = b = d$. Use of $MG = gR^2$ (from Exercise 13(iii)) then gives the result of Exercise 13(iv) once more.

Solutions to the exercises in Section 5

1. (i) If the orbit is bound, then we have

$$E = \frac{mMG}{2l}(e^2 - 1)$$

together with

$$l = \frac{b^2}{a} \quad \text{and} \quad e = \sqrt{1 - \frac{b^2}{a^2}}$$

(Equations (4) and (5) of Section 2). Hence $e^2 - 1 = -b^2/a^2 = -l/a$, giving

$$E = -\frac{mMG}{2a}.$$

(ii) For any orbit, Exercise 10(i) of Section 3 gives the distance of closest approach as

$$r_{\min} = \frac{l}{1+e}.$$

Hence we obtain

$$E = \frac{mMG}{2r_{\min}}(e - 1).$$

(As a check, both of these results agree with that of Exercise 12(iv) of Section 3 when the orbit is circular, with radius $d = a = r_{\min}$ and eccentricity $e = 0$.)

2. The result

$$d^3 = \frac{gR^2T^2}{4\pi^2}$$

can be obtained just as in parts (i)–(iv) of the solution to Exercise 13 in Section 3, though here d is the radius of the satellite's orbit rather than that of the Moon's. On putting $g = 9.81\,\mathrm{m\,s^{-2}}$, $R = 6.4 \times 10^6$ m and $T = 1\,\mathrm{day} = 24 \times 3600\,\mathrm{s}$, this gives

$$d^3 = \frac{1}{4\pi^2}\left(9.81 \times (6.4 \times 10^6)^2 \times (24 \times 3600)^2\right)$$

or $d \simeq 4.24 \times 10^7\ \mathrm{m} \simeq 6.6R$.

The corresponding value for the height above the Earth's surface is

$$d - R \simeq 3.6 \times 10^7\ \mathrm{m}.$$

(The use of geostationary orbits for communications satellites was first suggested by the science fiction author Arthur C. Clarke.)

3. (i) For bound motion, we have both

$$E = \tfrac{1}{2}mv^2 - \frac{GmM}{R}$$

and (from Exercise 1(i) of this section)

$$E = -\frac{mMG}{2a}.$$

On equating these expressions and solving for a, we obtain

$$a = \left(\frac{2}{R} - \frac{v^2}{MG}\right)^{-1}.$$

(ii) From Equations (4) and (5) of Section 2, we have

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{l}{a}}.$$

But also, it is known from Equation (16) of Section 3 that

$$l = \frac{L_z^2}{m^2MG}.$$

Since $L_z = mvR$ is given in the question, we have

$$l = \frac{v^2R^2}{MG}$$

and, using the result of part (i),

$$e = \left[1 - \frac{v^2R^2}{MG}\left(\frac{2}{R} - \frac{v^2}{MG}\right)\right]^{1/2}$$

$$= \left[1 - 2\left(\frac{v^2R}{MG}\right) + \left(\frac{v^2R}{MG}\right)^2\right]^{1/2}$$

or $e = \left|1 - \dfrac{v^2R}{MG}\right|$,

where the magnitude has been taken to ensure that $e \geq 0$. (Note that the condition $e < 1$ must correspond to the given condition $v^2 < 2MG/R$ for bound motion.)

Alternatively, this result could have been obtained from the formula

$$E = \frac{GmM}{2l}(e^2 - 1).$$

(iii) For circular motion, we have $e = 0$ and

$$v^2 = \frac{MG}{R}.$$

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